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R. P. Boas, Jr. J. L. Doob E. Hille

J. V. Wehausen, *Executive Editor*

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HISTORY

*Bunt, L. N. H. **Van Ahmes tot Euclides. Hoofdstukken uit de geschiedenis van de wiskunde.** [From Ahmes to Euclid. Chapters from the history of mathematics.] J. B. Wolters, Groningen-Djakarta, 1954. vii+171 pp. Paper, 4.90 florins; cloth, 5.50 florins.

This book is the result of an experiment conducted by the Section of Didactics of the Pedagogical Institute of the University of Utrecht. Recent reforms in the curriculum of the Netherlands gymnasia (Latin high schools) have made it possible to replace, in the so-called A-sections of these schools, most of the solid geometry by history of mathematics. The experiment consisted in sending out to a number of schools mimeographed material on the history of ancient mathematics, a field supposedly close to the interests of the students in these A-sections. The practice of teaching induced modifications and discussions. The outcome was the present school textbook on Greek and pre-Greek mathematics.

There are nine chapters. The first three of them deal with Egyptian and Babylonian mathematics, and the development of the positional system in the writing of numbers. The other chapters are introductions to the beginnings of Greek mathematics, Pythagoras and the Pythagoreans, Hippocrates and the classical three problems, Plato, Aristotle and Euclid. Stress is laid on a thorough training in the approach and in the methods of the ancients, so that the student who has followed the course has been obliged to work through such subjects as the Egyptian unit fraction calculus, the different approaches used by Hippocrates in his lunulae fragment, several trisection of angle problems and Book I of Euclid's Elements up to I, 47 (the theorem of Pythagoras). The text is in the Dutch language, but important places are in Greek and Dutch; the text of I, 47 is given in Greek alone. Selection and explanation is often guided by E. J. Dijksterhuis, De Elementen van Euclides [2 vols., Noordhoff, Groningen, 1929, 1930] and B. J. van der Waerden, Ontwakende wetenschap [Noordhoff, Groningen, 1950; English translation published by Noordhoff, 1954; these Rev. 12, 381; 16, 1]. There are many exercises, both in the text and at the end, such as: "Write 537 in Egyptian notation"; "Solve $5x^2-x=6$ in the Babylonian way"; "What is the nature of Pythagorean arithmetic?"; "What is the nature of the mathematical object according to Plato?"; "Sketch the structure of a definition according to Aristotle"; "Which is the first proposition in Euclid proved with the aid of the parallel postulate?". This book is therefore of interest to all who like to lift the history of mathematics out of the realm of the anecdotic into that of a science, and this already on the high school or college level. Further information can be found in the author's article in Acta Paedagogica Ultrajectina 6 (1954). *D. J. Struik.*

*Thomas of Bradwardine. **Tractatus de proportionibus. Its significance for the development of mathematical physics.** Edited and translated by H. Lamar Crosby, Jr. The University of Wisconsin Press, Madison, 1955. xi+203 pp. $3.50.

The translation of the text is preceded by an introduction (pp. 3–54) with a brief biography of Thomas of Bradwardine (c. 1290–1349), a discussion of texts and of the significance of the work, and an analysis of it. The translation itself (pp. 64–141) has the Latin text and translation on facing pages. Variant readings to the Latin text are also included (pp. 146–175). Notes and bibliography close the work.

Hofmann, J. E. Altes und Neues von der Quadratur des Descartesschen Blattes. Centaurus 3, 279–295 (1954).

*Molodšii, V. N. **Osnovy učeniya o čisle v XVIII veke.** [Elements of the study of number in the 18th century.] Gosudarstv. Učebno-Pedagog. Izdat., Moscow, 1953. 180 pp. 4.50 rubles.

Ch. 1. On mathematical rigor in the 18th century. Ch. 2. F. Engels on science, mathematics and metaphysics in the 17th and 18th centuries. Ch. 3. Views of mathematicians of the 18th century on the subject and the methods of the foundations of mathematics. Ch. 4. What is a number? Chapters 5–9 take up, respectively, questions of the foundations of the arithmetic of integers, fractions, irrationals, positive and negative numbers, and imaginary numbers.

*Taton, René. **L'histoire de la géométrie descriptive.** Les conférences du Palais de la Découverte. Université de Paris, Paris, 1954. 25 pp.

*Julia, Gaston. **Quelques réflexions sur les progrès comparés des mathématiques et de la physique en France après 1815.** Louis de Broglie, physicien et penseur, pp. 413–419. Editions Albin Michel, Paris, 1953. 870 francs.

*Kuratowski, Kazimierz. **Der Stand und die Aufgaben der Organisation des mathematischen Lebens in Volkspolen.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 1–9. Deutscher Verlag der Wissenschaften, Berlin, 1954.

*Infeld, L., gemeinsam mit R. Ingarden, M. Krzyzański, J. Rayski, W. Rubinowicz und W. Wrona. **Die Bedeutung der modernen Physik für die Entwicklung der Mathematik.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 95–109. Deutscher Verlag der Wissenschaften, Berlin, 1954.

*de Broglie, Louis. **Vue d'ensemble sur mes travaux scientifiques.** Louis de Broglie, physicien et penseur, pp. 457–493. Editions Albin Michel, Paris, 1953. 870 francs.

*Otradnyh, F. P. **Matematika XVIII veka i akademik Leonard Ėjler.** [Mathematics of the XVIIIth century and academician Leonhard Euler.] Gosudarstv. Izdat. "Sovetskaya Nauka", Moscow, 1954. 39 pp. .70 rubles.

Brief biography and discussion of Euler's contributions to various branches of mathematics.

*Worbs, Erich. **Carl Friedrich Gauss. Ein Lebensbild.** Koehler & Amelang, Leipzig, 1955. 236 pp. DM 7.50; $1.79.

*Otradnyh, F. P. **Mihail Vasil'evič Ostrogradskiĭ.** Izdat. Leningrad. Gos. Univ., Leningrad, 1953. 102 pp. (1 plate). 2.55 rubles.

*Pincherle, Salvatore. **Opere Scelte. Vol. II.** A cura della Unione Matematica Italiana. Edizioni Cremonese, Roma, 1954. 493 pp. 4500 Lire.

For vol. I see MR 15, 924. This volume contains the rest (23 papers) of the 38 papers selection for publication.

*Sonin, N. Ya. **Issledovaniya o cilindričeskih funkciyah i special'nyh polinomah.** [Investigations of cylinder functions and special polynomials.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 244 pp. (1 plate) 6 rubles.

Brief biography, list of Sonin's mathematical works, and the following selected papers (translated into Russian if necessary): Math. Ann. **16**, 1–80 (1880); Zapiski Novorossiĭsk. Obšč. Estest. **1885**, 1–8=Acta Math. **4**, 171–176 (1884); Math. Ann. **30**, 582–583 (1887); **59**, 529–552 (1904); Ann Sci. Ecole Norm. Sup. (3) **6**, 257–262 (1889); Mat. Sb. **14**, 527–536 (1890); Zapiski Imper. Akad. Nauk **69**, 1–30 (1892). There is also a commentary on the papers by the editor, N. I. Ahiezer, and also an essay: The works of N. Ya Sonin on approximate computation of definite integrals.

*Szász, Otto. **Collected mathematical papers.** Edited by H. D. Lipsich. Department of mathematics, University of Cincinnati, Cincinnati, Ohio; Hafner Publishing Co., New York, 1955. xiv + 1432 pp. (1 plate). $20.00.

Reproduction by photo-offset printing of 112 of O. Szász's mathematical papers. Omitted papers were either Hungarian papers which also appeared in a German version, book reviews, or monographs. The editor has also provided a brief biography. The papers are arranged according to field.

Sbrana, Francesco Il matematico pugliese Orazio Tedone. Confer. Sem. Mat.Univ. Bari no. 4, 17 pp. (1954).

FOUNDATIONS

Mostovskiĭ [Mostowski], A. The present state of investigations in the foundations of mathematics. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 3–38 (1954). (Russian)

This is an expanded version of a report read before the 8th Congress of Polish mathematicians in September 1953. The collaboration of the following persons is acknowledged: A. Grzegorczyk, J. Łoś, S. Mazur, H. Rasiowa, R. Sikorski, and S. Jaśkowski. It is principally a critical review of recent research in certain aspects of the foundations of mathematics. The aspects relate to what might be called applied foundations of mathematics, as opposed to foundations in the strictest sense. The main object is not to analyze the nature of mathematics and to exhibit it as constructed from simple elements, but to forge tools which can be used for manufacturing new mathematical results. Thus the author emphasizes those topics which connect logic with other parts of mathematics, discussing, to be sure, their significance, but always maintaining a strictly mathematical point of view. He treats in some detail the theory of axiomatic systems, based on the predicate calculus of first order, and their models; the theorems of completeness and incompleteness and the striking algebraic consequences obtained from the former; the mutual relations between logic and algebra generally; various constructive tendencies, especially the constructiveness principle invoked by Gödel in his tract on the continuum hypothesis; recursive analysis; decision problems; and the theory of recursive functions. Considerable space is devoted to the foundations of arithmetic and set theory; here the author interprets the incompleteness theorems as proving that the axiomatic point of view is untenable, but it is by no means clear what he intends to propose as an alternative.

The discussion is thought-provoking and penetrating. There are, to be sure, certain absurdities which appear to have been dictated by politics, and the reviewer thinks that the author is mistaken in his appraisal of the significance of formalization (the word 'metamathematics' does not occur, nor is there any indication of the connection between formalization and constructiveness). There is also some bias; e.g. in the discussion of recursive analysis the work of Goodstein is not mentioned. But in spite of these flaws the work seems to the reviewer a masterpiece of critical exposition. It contains also a statement (and even outline of proof) of certain recent results obtained by the collaborators some of which have not yet been published elsewhere. These include: results in "calculable analysis" by Mazur and Grzegorczyk; the proof of the completeness theorem by Rasiowa and Sikorski [Fund. Math. **37**, 193–200 (1950); MR **12**, 661], with indications of its extension to non-classical logics; and extensions due to Łoś of a theorem of G. Birkhoff concerning algebras which can be defined by equations. There are over a hundred references to the literature. *H. B. Curry* (University Park, Pa.).

Mostowski, Andrzej, in collaboration with A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa, and R. Sikorski. The present state of investigations on the foundations of mathematics. Rozprawy Mat. **9**, 48 pp. (1955).

English version of the paper reviewed above.

*Mostowski, A., **gemeinsam mit A. Grzegorczyk, S. Jaśkowski, J. Łos, S. Mazur, H. Rasiowa und R. Sikorski. Der gegenwärtige Stand der Grundlagenforschung in der Mathematik.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 11–44. Deutscher Verlag der Wissenschaften, Berlin, 1954.

German version of the paper reviewed second above.

Mostowski, A. The present state of investigations in the foundations of mathematics. Prace Mat. 1, 13–55 (1955). (Polish)

Polish version of the paper reviewed third above.

***Skolem, Th. Results in investigations in the foundations.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 273–289 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution). (Norwegian)

This is a survey of recent results on the foundations of mathematics as seen from the point of view of the author. The following topics are treated very sketchily: the foundations of set theory; the Löwenheim-Skolem theorem, and the relativism associated with it; impredicative definitions; intuitionistic critique; recursive number theory; and Post's theory of canonical syntactical systems. *H. B. Curry.*

Skolem, Th. The logical background of arithmetic. Bull. Soc. Math. Belg. 6 (1953), 23–34 (1954).

The author proposes to set up formal systems for arithmetic only after having established their consistency or conservative character with respect to numerical arithmetic (=correctness of all numerical formulae provable in the system considered). He confines himself to quantifier-free systems, and proposes to regard propositions involving quantifiers as incomplete communications regarding quantifier-free propositions; he proposes that a formula

$$(x_1)(Ey_1)\cdots(x_n)(Ey_n)R(x_1\cdots x_n, y_1\cdots y_n)$$

should always be interpreted as follows: there exist recursive functions $\phi_1(x)$, $\cdots$, $\phi_n(x_1, \cdots, x_n)$ such that

$$R[a_1\cdots a_n\phi_1(x_1)\cdots\phi_n(x_1\cdots x_n)],$$

where the a are free variables. The author prefers his proposal to Hilbert's consistency formulation because (i) the latter ignores the problem of interpreting the system studied, and (ii) "to prove consistency we must use considerations of a more difficult kind which seems to lead us into an infinite regress". The force of this criticism is not clear to the reviewer since (i) the author not only ignores the interpretation of the systems with quantifiers considered by Hilbert, such as Peano's arithmetic, but ignores the systems altogether, (ii) e.g. the consistency proof for Peano's arithmetic by Ackermann [Math. Ann. 117, 162–194 (1940); MR 1, 322] uses as "new" consideration ordinal induction up to the first ϵ-number which can be formulated in quantifier-free form [Kreisel, J. Symb. Logic 17, 43–58 (1952), para. 38; MR 14, 440]: on the author's own criteria this could not be called more "difficult" than the operations with quantifiers permitted in Peano's arithmetic. Further, even if one accepts the proposal to interpret quantified propositions in terms of quantifier-free ones, it seems unreasonable to demand the "naive" interpretation above; for, it is known that this interpretation is false for Peano's arithmetic, which, however, does possess an interpretation in quantifier-free arithmetic, namely the no-counter-example-interpretation given in the reviewer's paper cited above. The author does not seem to realize that the interpretation demanded by him applies to Peano's arithmetic when $n=1$; this too is ensured by the no-counter-example-interpretation.

The author also sketches Dedekind's development of arithmetic in set theory, and criticizes it because of its use of impredicative definitions: he calls a definition impredicative if "it defines an object which is required to be one of the values of a variable occurring in the defining expression". This does not seem a very convincing objection, since (on the author's definition) a μ-symbol definition $\mu_x(y)R(x, y)$ of an integer is impredicative; yet it can be used consistently.

There are some general remarks concerning intuitive reasoning including an interesting quotation from A. Thue. *G. Kreisel* (Reading).

Skolem, Th. Some considerations concerning recursive arithmetic. Bull. Soc. Math. Belg. 6 (1953), 35–46 (1954).

This paper contains the following, more or less isolated, results concerning the non-negative integers. (i) A simple construction of a recursively enumerable, but not recursive set of integers [cf. Skolem, Math. Scand. 1, 213–221 (1953); MR 15, 667]. (ii) A simpler base for Kalmar's class E of elementary functions generated by (superposition of) the functions 1, $x+y$, $|x-y|$, xy, $\sum_1^z$, $\prod_1^z$, namely: 1, $x+y$, $x\dot{-}y$ [$=\max(x-y, 0)$], x^y, $\sum_1^z$. (iii) Proof that x^y does not belong to the class H generated by the functions: 1, $x+y$, $x\dot{-}y$, $\sum_1^z$. (iv) Translation of certain equations $f(x)=y$ of E into equations $\rho(x, y)=0$ of H (even if f does not belong to H) such that $f(x)=y \leftrightarrow \rho(x, y)=0$, e.g. $x^2=y$ is so translated. If further $g(x)=y \leftrightarrow \sigma(x, y)=0$, then $f(x)+g(x)=y$, $f(x)\cdot g(x)=y$ can be translated into H; the proof depends essentially on the fact that if $f(x)+g(x)=y$ or $f(x)\cdot g(x)=y$ there exist numbers y_1, y_2, both $\leqq y$, such that $f(x)=y_1$, $g(x)=y_2$ ($y=0$ being a trivial exception). *G. Kreisel.*

***Schröter, Karl. Theorie des mathematischen Schliessens.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 5–12. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Expository article concerning (the following extension of) Gödel's completeness theorem for the predicate calculus of first order: if every finite subset of a (possibly nondenumerable) set M of formulae is consistent over the predicate calculus, then there exists a model of M. *G. Kreisel.*

***Lorenzen, Paul. Die Rolle der Logik in der Grundlagenkrisis der Analysis.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 65–73; discussion, pp. 73–74. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

An eloquent summary of the author's views on foundations. Modern criticism of classic analysis is reduced to two requirements: (a) we ought not to use all rules of logic when dealing with infinitely many individuals; (b) we ought not to apply the naive notion of a set. Then these negative requirements are turned into a positive program. (a) It is shown that, if in arithmetic we introduce certain predicates, we can define logical operations on expressions containing these predicates and then prove the familiar logical laws. However, a difficulty arises in connection with the principle of the excluded third. Under these circumstances, it remains a matter of convenience whether or not one decides to apply this principle. (b) We may obtain a constructive notion of a set by demanding that only sets definable in some constructed language S are admitted. Now we may start with an elementary "construct" S_0 and with the sets definable in S_0. This leads to enlarging S_0 into S_1 and hence to an extension of the notion of a set. In this way, we obtain a sequence of "constructs" S_0, S_1, S_2, $\cdots$, and even S_ω and $S_{\omega+1}$; and in $S_{\omega+1}$ we obtain a version of analysis which practically does not differ from classic analysis.

E. W. Beth (Amsterdam).

Mihăilescu, Eugen Gh. Investigations on subsystems of propositional calculus. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 1–44 (1951). (Romanian. Russian and French summaries)

This is a general report, with references to previous papers, on axiomatizations of subsystems of classical (i.e. two-valued) propositional algebra with various choices of primitive operations (or functors). The author uses the Łukasiewicz notation with operations (functors) as follows: E (equivalence), C (implication), A (alternation), K (conjunction), N (negation), R ("reciprocity", i.e. non-equivalence). He presents axiomatizations for systems admitting the following sets of primitive operations: (E), (E, N), (E, R), (E, R, N), (C), (C, A), (C, K), (C, A, E), (C, K, E), (C, A, K, E), (C, E, A, K, R, N), (E, A), (E, R, A). The system (E), for example, consists of the axioms $EEpqEqp$, $EEEpqrEpEqr$ due to Mihăilescu; the system (C) of the axiom $CCCpqCrsCtCCspCrp$ due to Łukasiewicz; the other systems are formed by adjoining to one or the other of these appropriate axioms for the new operations; while in all cases the rules are substitution and detachment (modus ponens) for C or E. In each case the object appears to give a formulation consistent with the classical algebra and sufficient to derive all theses of that algebra which contain only the operations in question; further to give normal forms from which a decision criterion can be obtained. The paper contains a general summary of results along these lines; in particular, all the positive systems are asserted to be complete, while certain ones involving N or R are not. There is no relation to intuitionistic or other similarly weakened systems. [For certain improvements see the review by H. Rasiowa in J. Symb. Logic **17**, 277–278 (1953).]

H. B. Curry (University Park, Pa.).

Prior, A. N. The interpretation of two systems of modal logic. J. Comput. Systems **1**, 201–208 (1954).

The author gives interpretations of the Lewis-von Wright and Łukasiewicz [same J. **1**, 111–149 (1953); MR **15**, 2] modal logics within the two-valued C-N-δ-$\prod$ propositional calculus. He then discusses applications to two Aristotelean theses and to differences between the work of Łukasiewicz and Parry [Parry, J. Symbolic Logic **4**, 137–154 (1939); MR **1**, 131]. *A. Rose* (Nottingham).

Anderson, Alan Ross. On the interpretation of a modal system of Łukasiewicz. J. Comput. Systems **1**, 209–210 (1954).

The author shows that the modal system of Łukasiewicz [same J. **1**, 111–149 (1953); MR **15**, 2] is semantically noncomplete in the sense of Halldén [J. Symb. Logic **16**, 127–129 (1951); MR **13**, 97]. *A. Rose* (Nottingham).

Anderson, Alan Ross. On alternative formulations of a modal system of Feys-von Wright. J. Comput. Systems **1**, 211–212 (1954).

The author gives a brief outline of a proof that von Wright's system M can be formalised using only substitution and modus ponens as primitive rules of procedure. He uses infinitely many axioms. Seven of these are stated explicitly and the remaining axioms are generated by means of the rule 'If α is an axiom, then $L\alpha$ is an axiom'. The note concludes with a discussion of applications to the Lewis systems of S4 and S5 and the author conjectures that the axioms of his formalisation cannot be replaced by a finite set of axioms. *A. Rose* (Nottingham).

Łukasiewicz, Jan. Arithmetic and modal logic. J. Comput. Systems **1**, 213–219 (1954).

The author considers the introduction of modal functors into the theory of equality based on the two axioms Θaa, $C\Theta abC\phi a\phi b$, where Θ denotes equality and ϕ is a proposition-forming functor of one numerical argument. He shows that this causes difficulties and therefore uses the Ł-modal system, [same J. **1**, 111–149 (1953); MR **15**, 2] in order to overcome them. He then applies the Ł-modal system to arithmetic in order to discuss the meaning of the so-called 'existential quantifier' and expresses the view that a formula of the form $\sum a\phi a$ should, in order for it to be expressed in words, be translated into the equivalent form $N\prod aN\phi a$.

A. Rose (Nottingham).

Tarski, Alfred. Contributions to the theory of models. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. **57**, 572–581, 582–588 (1954); **58**, 56–64 (1955) = Indagationes Math. **16**, 572–581, 582–588 (1954); **17**, 56–64 (1955).

This series of papers is concerned with the relations between the syntactical properties of given sentences (for the most part, in the lower predicate calculus) and the set-theoretical properties of the classes of models defined by them. Various substantial results are obtained, of which the following is typical. A sentence is called universal if it is in prenex normal form and includes none but universal quantifiers. A set of models ("structures", "systems") K is said to belong to the universal class if there exists a finite set of universal sentences which defines K. K is said to belong to the universal class in the wider sense if there exists a set of universal sentences which defines K. The author then proves that in order that a set of model K belong to the universal class in the wider sense, it is necessary and sufficient that the following three conditions be satisfied simultaneously. (i) If a model R belongs to K then all submodels (partial models) of R also belong to K. (ii) If R belongs to K then any model which is isomorphic to R also belongs to K. (iii) Any model R, which is such that all the finite submodels of R belong to K, must itself belong to K.

Similar theorems are given for sets of models which belong to the universal class (due to R. L. Vaught, see next review), or to certain relativised universal classes. Further results of this type are obtained concerning universally equivalent systems, equationally definable algebras and representable relation algebras. Prospective readers please note that the bibliography for the series under review is to be found at the end of the third paper. *A. Robinson* (Toronto, Ont.).

Vaught, Robert L. Remarks on universal classes of relational systems. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 589–591 (1954).

As a supplement to Tarski's theorem, which is quoted in the preceding review, the author proves the following. In order that a set of models K which includes only a finite number of relations belong to the universal class, it is necessary and sufficient that three particular conditions be satisfied simultaneously. Of these, two coincide with (i), (ii) in the preceding review, while the third condition requires the existence of an integer n, such that any model R which has the property that every submodel of R which contains no more than n elements belongs to K, must itself belong to K. An example is given to show that the restriction on the number of relations in R is essential. *A. Robinson.*

Chang, Chen-Chung. Some general theorems on direct products and their applications in the theory of models. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 592–598 (1954).

This paper is concerned with the set-theoretical characterisation of equationally definable algebras, where the term "algebra" is to be understood in the sense of G. Birkhoff [Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)]. A typical result is the following. In order that a class of algebras be equationally definable in the wider sense, it is necessary and sufficient that the following three conditions be satisfied simultaneously. (i) If A is an algebra, and if all the subsystems of A whose cardinal is smaller than γ, are subsystems of elements of K, then A belongs to K. (ii) Any homomorphic image of an element of K which is an algebra, also belongs to K. (iii) Let A be a direct product of elements of K such that the order of the product is smaller than γ. Then A belongs to K.

In this theorem, γ is any cardinal which is greater than the number of relations which occur in the algebras of K. *A. Robinson* (Toronto, Ont.).

Valpola, Veli. Elementare Untersuchungen der Antinomien von Russell, Grelling-Nelson und Eubulides. Theoria **19**, 183–188 (1953).

In the case of each of the antinomies mentioned the author derives from "ganz evidenten Grundannahmen", using the symbolism of the predicate calculus, the non-existence of some notion associated with the paradox. These conclusions are not new, but it is perhaps unusual to see them so obtained. There follows a "Schluss" in which the reviewer is unable to find any conclusions of value. *H. B. Curry.*

Zubieta Russi, Gonzalo. Some theorems in the theory of elementary quantification. Bol. Soc. Mat. Mexicana **8**, 33–46 (1951). (Spanish)

The author gives a formulation of the predicate calculus of first order in terms of the stroke function and universal quantification as primitive operations. He then proves a series of theorems concerning that calculus. These conclude with the completeness theorem, which is proved elegantly by a method similar to Henkin's [J. Symb. Logic **14**, 159–166 (1949); MR **11**, 487]. *H. B. Curry.*

Zubieta R., F. On a normal notion of the theory of classes. Bol. Soc. Mat. Mexicana **10**, nos. 3–4, 33–34 (1953). (Spanish)

The author suggests the following definition of uniformity of a relation X with respect to a relation Y:

$$\mathrm{Un}(Y, X) = (x)(y)(u)(v)[\langle ux\rangle \varepsilon X, \langle vy\rangle \varepsilon X . \supset . \langle xy\rangle \varepsilon Y \supset \langle uv\rangle \varepsilon Y].$$

A function is then a relation which is uniform with respect to identity. *H. B. Curry* (University Park, Pa.).

Routledge, N. A. Concerning definable sets. Fund. Math. **41**, 6–11 (1954).

The author proves that [in the notation of Mostowski, Fund. Math. **34**, 81–112 (1947); MR **9**, 129] $P_n^{(k)} \cap Q_n^{(k)}$ is the class of all sets Turing derivable from [Turing reducible to, in the language of Post, Bull. Amer. Math. Soc. **50**, 284–316 (1944); MR **6**, 29] a set of $P_{n-1}^{(1)}$. The proof makes use of primitive recursive functions and Turing machines. The author points out that the same result is contained in Theorem XI, page 293 of Kleene's "Introduction to metamathematics" [Van Nostrand, New York, 1952; MR **14**, 525] which appeared after the present paper was written. *H. G. Rice* (Durham, N. H.).

Freudenthal, H. Les possibilités des machines à calculer. Bull. Soc. Math. Belgique **6** (1953), 14–22 (1954).

This expository paper presents in simplified and neat form the Turing-Post theory of computable numbers [A. M. Turing, Proc. London Math. Soc. (2) **42**, 230–265 (1936); E. L. Post, J. Symb. Logic **1**, 103–105 (1936); **12**, 1–11 (1947); MR **8**, 558]. The main simplification is in a binary coded description number for special Turing machines, similar to the "Post number", instead of Turing's "description number" or the "Gödel number" as defined in S. C. Kleene, "Introduction to metamathematics" [Van Nostrand, New York, 1952, Chap. XIII; MR **14**, 525]. The exposition describes special Turing machines, shows how to enumerate computable numbers, gives Turing's argument using the Cantor diagonal method to prove the existence of a non-computable number, remarks on the existence of a universal Turing machine, and gives Turing's argument to resolve the apparent paradox. Finally, the relationship with Gödel's theorem is discussed. On page 18 the number 2,504,681 should read 2,504,301. *S. Gorn* (Aberdeen, Md.).

Henry, D. P. Expressions trivially decidable. J. Comput. Systems **1**, 221–224 (1954).

Meserve, B. E. Decision methods for elementary algebra. Amer. Math. Monthly **62**, 1–8 (1955).

Expository article on the Sturm-Tarski decision method for elementary algebra. *G. Kreisel* (Reading).

Lacombe, Daniel. Sur le semi-réseau constitué par les degrés d'indécidabilité récursive. C. R. Acad. Sci. Paris **239**, 1108–1109 (1954).

Verf. kündigt zwei Sätze über den Halbverband der rekursiven Unentscheidbarkeitsgrade [Kleene und Post, Ann. of Math. (2) **59**, 379–407 (1954); MR **15**, 772] an, aus denen insbesondere folgt, dass die arithmetischen Unentscheidbarkeitsgrade keinen Verband bilden. *P. Lorenzen.*

Takekuma, Ryōichi. On a nine-valued propositional calculus. J. Comput. Systems **1**, 225–228 (1954).

The author first gives lattice-theoretic definitions of Kuroda's logic and weakly Brouwerian logic. He then shows that a logic whose truth-values form a certain nine-element lattice is not Kuroda's logic but is weakly Brouwerian. *A. Rose* (Nottingham).

***Tamari, Dov. Some mutual applications of logic and mathematics.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 89–90. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

The author first develops algebraic methods of finding functionally complete sets of primitives for m-valued logics. He then discusses very briefly the use of logical notation in mathematical proofs and the arithmetic of grouping symbols for binary operations. *A. Rose* (Nottingham).

✱**Heyting, A. Logique et intuitionnisme.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 75–82; discussion, pp. 82–83. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

The author discusses the differences between formalist and intuitionist methods of proof, considering several examples. He feels that metamathematics can be regarded as a chapter of intuitionist mathematics. He finds interesting the theorems of others concerning the replacement of intuitionism by formal interpretations, but feels that intuitionism ought not to be so replaced. A formal system cannot be an intuitionist system. *A. Rose.*

Rasiowa, H. Constructive theories. Bull. Acad. Polon. Sci. Cl. III. **2**, 121–124 (1954).

Ankündigung eines topologischen Kriteriums für die Konstruktivität einer intuitionistisch formalisierten Theorie T. T heisst konstruktiv, wenn gilt: (1) Ist $\alpha \vee \beta$ ein Theorem von T, dann ist auch α oder β ein Theorem von T. (2) Ist $\sum_x \alpha(x)$ ein Theorem von T, dann gibt es einen Term ξ von T, so dass $\alpha(\xi)$ ein Theorem von T. Für das Kriterium wird der Brouwersche Verband der Klassen in T äquivalenter Formeln isomorph auf die abgeschlossenen Untermengen eines topologischen Raumes abgebildet. Als Folgerungen des Kriteriums werden u. a. angekündigt: ein Analogon des Herbrandschen Theorems für konstruktive Theorien und die Konstruktivität aller Theorien mit disjunktionsfreien Axiomen. Die Beweise werden später erscheinen.

P. Lorenzen (Bonn).

✱**Beth, E. W. Observations métamathématiques sur les structures simplement ordonnées.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 29–35; discussion, p. 35. Gauthier-Villars, Paris: E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Während meist nur formalisierte Theorien, deren Axiomensysteme aus Gleichungen bestehen (Birkhoffsche Theorien) metamathematisch betrachtet werden, untersucht Verf. die totalen Ordnungen $(S, <)$. Mit $a+b=\min(a, b)$ lassen sich diese als kommutative Halbgruppen $(S, +)$ charakterisieren, die $x+y \in \{x, y\}$ erfüllen. Die Theorie der totalen Ordnungen ist nicht Birkhoffsch, da das direkte Produkt zweier Modelle nicht allemal wieder ein Modell ist. Für zwei Modelle $\mathfrak{S}_1$ und $\mathfrak{S}_2$ lässt sich jedoch stets die geordnete Summe bilden. Diese ist genau dann entscheidbar, wenn $\mathfrak{S}_1$ und $\mathfrak{S}_2$ entscheidbar sind. In Anschluss an Tarski wird jedem Modell $\mathfrak{S}$ die Menge der von $\mathfrak{S}$ erfüllen Aussagen als "Eudoxischer Typ" zugeordnet. Modelle gleichen Typs sind nicht allemal isomorph. Verf. erörtert den Zusammenhang seiner Untersuchungen mit denen Tarskis über Boolesche Verbände auf Grund des Satzes, dass jeder (abzählbare) Boolesche Verband isomorph zum Verband der Intervalle $(a, b]$ einer total geordneten Menge ist.

P. Lorenzen (Bonn).

✱**Rose, Alan. Caractérisation, au moyen de la théorie des treillis, du calcul de propositions à foncteurs variables.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 87–88. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Dem Aussagenkalkül entspricht algebraisch die Theorie der Booleschen Verbände. Verf. gibt an, wie diese Entsprechung auszudehnen ist auf die Lesniewskische Erweiterung des Aussagenkalküls durch einen variablen Aussagenfunktor. *P. Lorenzen* (Bonn).

Bernays, Paul. A system of axiomatic set theory. VII. J. Symb. Logic **19**, 81–96 (1954).

This discusses the construction of three independence models $\mathfrak{M}$, $\mathfrak{M}_1$, and $\mathfrak{M}_2$, which were announced in previous installments of this series [same J. **2**, 65–77 (1937); **6**, 1–17 (1941); **7**, 65–89, 133–145 (1942); **8**, 89–106 (1943); **13**, 65–79 (1948); MR **2**, 210; **3**, 290; **4**, 183; **5**, 198; **10**, 3], in particular in part VI. In all three models the elements are natural numbers of the basic system, with special number-theoretic definitions of the relation which is to play the role of the element relation ϵ.

The author discusses also the reduction of his basic set theory, consisting of axioms I, II, III, and VII, to a number-theoretic frame, allowing the application of known constructive consistency proofs of number theory. To take care of complications caused by the presence of two membership relations ϵ and η for sets and classes, he adapts methods recently introduced by I. Novak, G. Hasenjäger, and L. Henkin. He shows also the redundancy of one of his original axioms, namely IIIb(1), asserting the existence of the class of all unit sets, and discusses further the three independence models he constructed in part VI. *O. Frink.*

✱**Polya, G. Induction and analogy in mathematics. Mathematics and plausible reasoning, vol. I.** Princeton University Press, Princeton, N. J., 1954. xvi+280 pp. $5.50.

This is a sequel to the same author's "How to solve it" [Princeton, 1945, 1948; MR **6**, 198; **9**, 488]. A great number of interesting examples from various parts of mathematics are discussed in order to point out the way which leads from a problem to its solution and, more generally, the manner in which mathematical research should be conducted; in connection with these examples, a number of "loci", which more or less vaguely show the right direction (such as: generalisation, specialisation, analogy), and of devices (mathematical tools which can be applied in a straightforward fashion: mathematical induction, generating functions) are explained. In this regard, the reviewer would like to suggest the following additions: (1) "strong" or "course-of-values" induction, (2) looking for an indirect proof, and (3) considering various definitions of the notions involved.

E. W. Beth (Amsterdam).

✱**Polya, G. Patterns of plausible inference. Mathematics and plausible reasoning, vol. II.** Princeton University Press, Princeton, N. J., 1954. x+190 pp. $4.50.

The companion volume (cf. the preceding review) in the first place expounds a number of "heuristic patterns", such as:

A implies B
B without A hardly credible
B true

A very much more credible

and then applies these patterns, for instance, in deciding between rival conjectures. Then the theory of probability and its connections with the logic of plausible reasoning are discussed; the final chapter is devoted to "plausible reasoning in invention and instruction". An appraisal of the

author's (valuable) contributions must, of course, take into account the fact that it is mainly the outcome of one man's efforts [although O. Hölder, Die mathematische Methode, Springer, Berlin, 1924, can be mentioned as a forerunner]. It seems strange that (in particular, modern) logic receives so little attention. For instance, such material as is contained in Löwenheim, Scripta Math. **12**, 125–139 (1946) [MR **8**, 307] could certainly be incorporated into a logic of plausible inference. *E. W. Beth* (Amsterdam).

* **Tonnelat, Marie-Antoinette. La notion de réalité physique et l'oeuvre de M. Louis de Broglie.** Louis de Broglie, physicien et penseur, pp. 145–152. Editions Albin Michel, Paris, 1953. 870 francs.

ALGEBRA

Moser, Leo, and Wyman, Max. On an array of Aitken. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 31–37 (1954).

A. C. Aitken's array [Edinburgh Math. Notes **28**, 18–23 (1933)] is a double-entry table used to calculate numbers, G_n, by successive additions. The numbers G_n, which have many combinatorial interpretations, may be defined by $G_0=1$, $G_{n+1}=(G+1)^n$, if the last is understood in the usual symbolic sense: $G^n \equiv G_n$. The authors define a new set of numbers B_n by similar equations, namely $B_0=0$, $B_1=1$, $B_{n+1}=(B+1)^n$, $n>1$, and determine the exponential generating function

$$\sum B_n x^n/n! = \exp(e^x)\int_0^x \exp(-e^t)dt,$$

and the explicit expression $B_n=\sum S(r,n)\alpha_{r-1}$, where $S(r,n)$ is a Stirling number of the second kind, and

$$\alpha_n = 0!-1!+2!-\cdots+(-1)^n n!.$$

They also notice that, for p an odd prime

$$B_p \equiv 1+\alpha_{p-1} \pmod p$$
$$\equiv -(1+1/2!+\cdots+1/(p-1)!) \pmod p.$$

J. Riordan (New York, N. Y.).

Walls, Nancy. On a certain type of space-tableau. Proc. Edinburgh Math. Soc. (2) **9**, 82–86 (1954).

The notion of a standard Young tableau is used to define a space tableau, but no significance is attached to it.

G. de B. Robinson (Toronto, Ont.).

Braumann, Pedro Bruno Teodoro. A new method for obtaining an old formula. Ciência **4**, nos. 9–10, 68–71 (1954). (Portuguese. German summary)

This paper gives a new proof of a combinatorial formula by Mac Mahon by means of an expansion for cyclic determinants given by the reviewer [Duke Math. J. **18**, 343–354 (1951); MR **13**, 98]. *O. Ore* (New Haven, Conn.).

Rodeja F., E. G.-. Note on determinants of hyperbolic sines and cosines. Revista Mat. Hisp.-Amer. (4) **14**, 200–201 (1954). (Spanish)

Let $k_1, k_2, \cdots, k_n$ be integers, $0 \leqq k_1 < k_2 < \cdots < k_n$, and let

$$\mathrm{Ch}(k_1, k_2, \cdots, k_n) = \begin{vmatrix} \cosh k_1\theta_1 & \cosh k_1\theta_2 \cdots \cosh k_1\theta_n \\ \cosh k_2\theta_1 & \cosh k_2\theta_2 \cdots \cosh k_2\theta_n \\ \cdots & \cdots \\ \cosh k_n\theta_1 & \cosh k_n\theta_2 \cdots \cosh k_n\theta_n \end{vmatrix}.$$

It is proved that for all real θ_j,

$$\frac{\mathrm{Ch}(k_1, k_2, \cdots, k_n)}{\mathrm{Ch}(0, 1, \cdots, n-1)} > \frac{\prod_{k>j-1}^{n}(k_k^2-k_j^2)}{(n-1)!1!3!5!\cdots(2n-3)!}$$

and this lower bound is best possible. A similar sharp result is obtained for ratios of determinants in $\sinh k_j\theta_j$.

A. W. Goodman (Lexington, Ky.).

Rodeja F., E. G.-. Note on determinants of hiperbolic sines and cosines. Revista Mat. Hisp.-Amer. (4) **14**, 202–203 (1954).

A translation into English of the paper reviewed above.

A. W. Goodman (Lexington, Ky.).

Burchnall, J. L. A method of evaluating certain determinants. Proc. Edinburgh Math. Soc. (2) **9**, 100–104 (1954).

For the determinant $|a_{st}|$ $(s, t=0, 1, \cdots, n)$, suppose that the corresponding bilinear form has been expressed as

$$(*) \qquad \sum_{s=0}^{n}\sum_{t=0}^{n} a_{st}X_sY_t = \sum_{r=0}^{n} c_r\left\{\sum_{s=r}^{n} p_{rs}X_s\right\}\left\{\sum_{t=r}^{n} q_{rt}Y_t\right\}$$

with $p_{rr}=q_{rr}=1$ for all r. Then $|a_{st}| = \prod_{r=0}^{n} c_r$. Now (*) implies and is implied by the identities $a_{st}=\sum_{r=0}^{\min(s,t)} c_r p_{rs} q_{rt}$, so that from any known identity of this form, with $p_{rr}=q_{rr}=1$, the value of the corresponding determinant can be written down at once. Applications are made to persymmetric determinants of Hermite polynomials, Legendre polynomials, etc. *E. F. Beckenbach* (Los Angeles, Calif.).

Williamson, J. H. The characteristic polynomials of AB and BA. Edinburgh Math. Notes no. **39**, 13 (1954).

The fact that the characteristic polynomials of AB and BA are identical follows from the principle of irrelevance of algebraic identities. A neat form of the argument is to consider the matrix $(A-\mu I)B(A-\mu I)-\lambda(A-\mu I)$, the determinant of which is equal on the one hand to

$$\det(A-\mu I)\cdot\det[B(A-\mu I)-\lambda],$$

and on the other hand to $\det[(A-\mu I)B-\lambda]\cdot\det(A-\mu I)$. Since the equality between these last two polynomials is an identity, it holds also with μ equal to 0.

J. L. Brenner (Pullman, Wash.).

Marcus, M. D. A remark on a norm inequality for square matrices. Proc. Amer. Math. Soc. **6**, 117–119 (1955).

Let A be a non-nilpotent square matrix; let $g(z)=a_n z^n$ be a power series such that all characteristic roots of A are contained within its circle of convergence. If the Jordan canonical form TAT^{-1} of A has exactly α boxes of dimensions k_i $(i=1, \cdots, \alpha)$ and with respective characteristic roots λ_i, the inequality

$$\left(\sum_{j=1}^{n}|g(\lambda_j)|^2\right)^{1/2} \leqq \|g(A)\| \leqq c\left(\sum_{j=1}^{\alpha}\sum_{s=0}^{k_j-1}\frac{(k_j-s)}{(s!)^2}|g^{(s)}(\lambda_j)|^2\right)^{1/2}$$

is obtained. The norm function involved is $(\mathrm{tr}\,AA^*)^{1/2}=\|A\|$. This is shown to reduce to a result of Gautschi [Duke Math. J. **20**, 127–140 (1953); MR **15**, 94] in case $g(z)$ is z^p. As an application, a theorem is proved on the asymptotic behavior of solutions of the system $\dot{x}=Ax$ with constant coefficients.

J. L. Brenner (Pullman, Wash.).

***Wong, Y. K. Inequalities for Minkowski-Leontief matrices.** Economic activity analysis, pp. 201–281. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London. $6.75.

The author is concerned with developing inequalities for matrices $A=(a_{ij})$. In part I an algebra norm $\|A\|$ is introduced which satisfies the usual axioms and, in addition, for A, B consisting of non-negative elements:

$$\|A+B\| \geqq \max(\|A\|, \|B\|).$$

The results of part I are essentially all known and consist of summarizing some of the usual simple properties of norms in Banach algebras. Part II deals with inequalities for matrices A which satisfy $\|A^q\|^{1/q}<1$ and consist of non-negative elements (Minkowski-Leontief matrices). Bounds are obtained for determinants of $I-A$. Some typical results in this direction are as follows: (1) If d_k is the determinant of the principal submatrix consisting of the first k rows and columns of $I-A$, then $d_n \leqq d_{n-1} \leqq \cdots \leqq d_1 = 1-a_{11}$ and $d_k < d_{k-1}$ if $a_{kk}>0$. It follows that all the principal minors of $I-A$ are necessarily positive if $I-A$ is non-singular. (2) If $c_j = |a_j| + s_j \leqq 1$, where $s_j = \sum_i |a_{ij}|$, then $\sum |a_i|\,|b_{ij}| \leqq c_j|d|$, where $B=(b_{ij})$ is the adjoint of $I-A$ and $d=\det B$. Numerous other inequalities of the same nature are given.

S. Karlin (Pasadena, Calif.).

***Wong, Y. K. Some mathematical concepts for linear economic models.** Economic activity analysis, pp. 283–339. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London. $6.75.

Several mathematical quantities such as the numerical range of a matrix $(x'Ax)$, quasi-inverse of a matrix, the convexity of Minkowski-Leontief matrices, etc. are given economic interpretations. The results are all essentially immediate or well known. However, the paper serves a purpose in acquainting the economist with relevant mathematical concepts and tools. *S. Karlin* (Pasadena, Calif.).

Wong, Y. K. Quasi-inverses associated with Minkowski-Leontief matrices. Econometrica **22**, 350–359 (1954).

This note studies the affect of errors on the inverses of matrices of the Minkowski-Leontief type. Except for slight changes, it summarizes work of the author, to be found in the two papers reviewed above and in Proc. Nat. Acad. Sci. U. S. A. **40**, 121–124 (1954); MR **15**, 926.

H. W. Kuhn (Bryn Mawr, Pa.).

Haynsworth, Emilie V. Quasi-stochastic matrices. Duke Math. J. **22**, 15–24 (1955).

In this paper, the assumption that a fixed vector is a characteristic vector of the matrix A is shown to lead to information on the characteristic roots of the matrix. The matrix A is called q.-s. (quasi-stochastic) [generalized q.-s.] if the relation $Av=v$ [$Av=vS'$, S' scalar] holds,

$$v=(1, 1, \cdots, 1_k, p, p, \cdots, p)^*.$$

If such a matrix can be partitioned $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$, where Q, R have k rows and S, T have $n-k$ rows, with respective row-sums $\begin{pmatrix} q & r \\ s & t \end{pmatrix}$, $rs \neq 0$, then it is q.-s. for two values of p, and thus two characteristic roots can be found. The other $n-2$ roots can be obtained by solving a certain matrix of order $n-2$. If x is real, the locus of two roots of $\begin{pmatrix} Q-xI & R \\ S & T \end{pmatrix}$ is a hyperbola. When p is real and exceeds 1, the dominant root of a q.-s. matrix A lies outside each of the first k Geršgorin circles $|\lambda - a_{ii}| \leqq \sum_j' |a_{ij}|$ and inside each of the others. If p is real and A is normal ($AA^*=A^*A$), then A is doubly q.-s., i.e. A^* is also q.-s. *J. L. Brenner.*

Ostrowski, A. M. On the spectrum of a one-parametric family of matrices. J. Reine Angew. Math. **193**, 143–160 (1954).

Let $D(n, \alpha)$ be the triangular $n \times n$ matrix with all entries on the main diagonal equal to α (real), and all $\frac{1}{2}(n^2-n)$ entries below the main diagonal equal to 1 ($n>1$). The spectrum of D^*D is studied. Let $\lambda^2(n, \alpha)$, $\Lambda^2(n, \alpha)$ be respectively the minimum and maximum characteristic roots of D^*D. For $\alpha=0$, $\frac{1}{2}$, 1 respectively, λ is 0, $\frac{1}{2}\tan \pi/4n$, $\frac{1}{2}\sec \pi/(2n+1)$; Λ is

$$\tfrac{1}{2}\csc \pi/(4n-2), \quad \tfrac{1}{2}\cot \pi/4n, \quad \tfrac{1}{2}\csc \pi/(4n+2).$$

Asymptotic formulas are given for λ, Λ. Recursion formulas appear for the nth degree polynomial which has root λ. λ increases monotonically as α decreases from 0.

J. L. Brenner (Pullman, Wash.).

Morinaga, Kakutaro, and Nôno, Takayuki. On the non-commutative solutions of the exponential equation $e^xe^y=e^{x+y}$. J. Sci. Hiroshima Univ. Ser. A. **17**, 345–358 (1954).

This paper obtains and classifies (independently) the 2×2 matrix solutions x, y of $e^xe^y=e^{x+y}$ and of $e^xe^y=e^ye^x$ when the base field is the field of complex numbers. Complete results are also obtained in case x, y come from a complex ternary algebra of degree two, or from the algebra of quaternions. More structural theorems (i.e. equivalences) and formulas connecting the solutions are given than in the papers of Huff [Rend. Circ. Mat. Palermo (2) **2**, 326–330 (1954); MR **16**, 4], Kakar [ibid. **2**, 331–345 (1954); MR **16**, 4], and G. N. Čebotarev [Dokl. Akad. Nauk SSSR (N.S.) **96**, 1109–1112 (1954); MR **16**, 4]. *J. L. Brenner.*

Šteĭnberg, A. S. On the problem of best uniform approximation for systems of linear incompatible equations and on the method of equation-gradient corrections. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 167–173 (1952). (Ukrainian. Russian summary)

The "Fourier descent method" [Oeuvres, t. II, Gauthier-Villars, Paris, 1890, pp. 315–328; Analyse des équations déterminées, partie I, Firmin Didot, Paris, 1831, p. 81] for finding the lowest point of a polyhedron (viz. from an interior point descending vertically to a face, then in negative gradient direction to a lower-dimensional face, etc., until a vertex is reached, then descending along edges) is carried through in the special case of minimizing the Chebyshev deviation (=maximum absolute left-hand side) for an unsolvable system of linear equations. The procedure is far from uniquely determined, since a choice of descending edges and, on any face not of highest dimension, of gradient-defining functions, is allowed. In the de la Vallée Poussin (i.e., non-degenerate) case, the method was evolved by Zuhovickiĭ [Dokl. Akad. Nauk SSSR (N.S.) **79**, 561–564 (1951); MR **13**, 285; dissertation, Kiev, 1950, part II].

T. S. Motzkin (Los Angeles, Calif.).

Zuhovic'kiĭ, S. I. An algorithm for finding the point of least deviation (in the sense of P. L. Čebyšev) from a given system of m points. Dopovidi Akad. Nauk Ukrain. RSR 1951, 404–407 (1951). (Ukrainian. Russian summary)

The center of the smallest sphere containing m given points is found by a procedure analogous to the one described in the preceding review. *T. S. Motzkin.*

Krull, Wolfgang. Über Polynomzerlegung mit endlich vielen Schritten. III. Math. Z. 60, 109–111 (1954).

A sequence $K_1, K_2, \cdots$ of algebraic number fields has "property (Z)" if for each irreducible integral polynomial f there exists a computable bound ν^* such that f is irreducible over every K_i provided f is irreducible over $K_1, \cdots, K_{\nu^*}$ [Krull, Math. Z. 59, 57–60, 296–298 (1953); MR 15, 96, 508]. The author gives two criteria for property (Z) and discusses the connection between them. The "D-criterion" (credited to M. Kneser, unpublished) is the following: for each integer $D \neq 0, 1$ there exists a computable bound ν_0 such that no K_i with $i > \nu_0$ has a subfield of discriminant D. The "A-criterion" involves the existence of certain polynomials φ_{in}, and for each pair (n, M) of integers >1 the computability of a bound ν, such that if $\mu > \nu$ then an irreducible integral polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ is irreducible over K_μ provided $|\varphi_{in}(a_1, \cdots, a_n)| \leq M$ for every i.

E. R. Kolchin (Paris).

Abstract Algebra

***Rédei, László. Algebra. I. Kötet. [Algebra. Vol. I.]** Akadémiai Kiadó, Budapest, 1954. 642 pp. 110 ft.

The book is intended to be both a university text and a reference volume. It treats its subject carefully, systematically, and exhaustively. The exposition is, in principle, self-contained. There are no exercises, but there are many examples and counterexamples designed to supply the concrete background needed for the understanding of the abstract material. Whenever possible, the author proceeds from the general to the special.

Since the language of the book is Hungarian, so that the set of its readers has an almost unsurmountable natural boundary, and since, also, what is new in the book is more in the exposition, arrangement, and technical detail than in the content, this review is merely descriptive. What follows is a list, by chapters, of the principal topics that are considered.

I. Set theory (non-axiomatic); several forms of Zorn's lemma. II. The concept of an algebraic structure (a set equipped with some operations, usually binary), semigroups, rings, skew fields, homomorphism, quotient with respect to an equivalence relation, the Jordan-Hölder-Schreier theorem. III. Structures with operators, universal algebraic considerations, free structures, polynomial rings, determinants, quaternions. IV. Euclidean rings, ideals, divisibility, Euclidean algorithm. V. Finite abelian groups, the fundamental theorem; Hajos' theorem. VI. Modules, vector spaces, matrices, elementary divisors, finitely generated abelian groups. VII. Rings of polynomials, zero divisors, derivatives, multiple factors, symmetric polynomials, interpolation, Eisenstein's theorem, ideals in commutative rings. VIII. Field theory, extensions, normality, cyclotomy, finite fields, Wedderburn's theorem, transcendental extensions, separable extensions, norm and trace. IX. Ordered structures, archimedean order, absolute value. X. Fields with valuations, real numbers, real closed fields, non-archimedean valuations, Ostrowski's theorem, the Hensel lemma. XI. Galois theory, quadratic reciprocity, cyclic fields, solvability, the general equation, solution of cubic and quartic equations, geometric constructions, the normal basis theorem. *P. R. Halmos* (Chicago, Ill.).

Stone, M. H. Free Boolean rings and algebras. An. Acad. Brasil. Ci. 26, 9–17 (1954).

Verf. führt in Verallgemeinerung der Gruppenalgebren eine Konstruktion von Algebren für multiplikative Systeme über Ringen durch. Als Spezialfall ist darin eine Konstruktion der freien Booleschen Ringe enthalten. Die Zusammenhänge mit der universellen Algebra und der Aussagenlogik werden erörtert. *P. Lorenzen* (Bonn).

***Zappa, Guido. Reticoli e geometrie finite.** Lezioni raccolte dal dott. Giovanni Zacher. Libreria Editrice Liguori, Napoli, 1952. 256 pp. 1300 Lire.

The chapter headings in this book are as follows. 1. Generalities on lattices. 2. Semimodular lattices. 3. Modular lattices. 4. Complemented modular lattices. 5. Projective spaces. 6. Finite Desarguesian projective planes. There is also an appendix dealing with automorphisms of elementary abelian groups.

A projective geometry is defined in terms of lattice properties, and for Desarguesian geometries, representation is in terms of the collineation group rather than in terms of more conventional coordinate systems.

The first chapter includes material on homomorphisms, ideals, direct products, and the center of a lattice. The second and third chapters cover the Jordan-Hölder theorems, independent elements, and projectivities in modular lattices. The fourth chapter leads up to the major theorem that every complemented modular lattice with the ascending chain condition is the direct product of simple lattices, these being projective geometries. In the fifth chapter projective geometries are treated as simple complemented modular lattices, and affine spaces as semimodular lattices. The Theorem of Desargues is proved in spaces of dimension higher than that of a plane. The study of finite Desarguesian planes is based on homologies, and these lead to a representation of the plane in terms of an elementary abelian group and its automorphisms. *Marshall Hall, Jr.* (Columbus, Ohio).

Gomes, Alvércio M. Completion by cuts of a distributive lattice. Rev. Científica 3, no. 3–4, 56–68 (1952).

The author considers the problem of necessary and sufficient conditions that the completion by cuts of a distributive lattice be distributive. This paper considers the case where the given distributive lattice is a sub-lattice of the product of two chains. The significance of this case is left in some doubt as the example cited by the author to show that, in the case of more than two chains, the completion by cuts is not necessarily distributive is ineffective.

H. M. MacNeille (St. Louis, Mo.).

Thurston, A.H. Congruences on a distributive lattice. Proc. Edinburgh Math. Soc. (2) 10, 76–77 (1954).

Let L be a lattice of finite subsets of a set S, and q a congruence relation on L. Then there is a subset K of S such that $X \equiv Y(q)$ if and only if $X \cap K = Y \cap K$. Any two congruences on L are permutable. Similar theorems for other kinds of lattices are cited; cf. G. Birkhoff, Lattice

theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 119; MR 10, 673; R. P. Dilworth, Ann. of Math. (2) 51 [not 50, as given in the paper], 348–359 (1950); MR 11, 489. *P. M. Whitman.*

Pinsker, A. G. Lattices equivalent to K-spaces. Dokl. Akad. Nauk SSSR (N.S.) 99, 503–505 (1954). (Russian)

Complicated necessary and sufficient conditions are announced, without proofs, for a lattice to be isomorphic with the lattice of a complete lattice-ordered vector space with unit over the real field. *E. Hewitt* (Seattle, Wash.).

Nakayama, Tadasi. Generalized Galois theory for rings with minimum condition. II. Amer. J. Math. 77, 1–16 (1955).

[Pour la partie I voir même J. 73, 1–12 (1951); MR 12, 668.] L'auteur poursuit ses recherches sur l'extension de la théorie de Galois aux anneaux primaires ayant un élément unité et satisfaisant à la condition minimale. Pour certains groupes d'automorphismes Φ d'un tel anneau R, dits "réguliers", il montre qu'il y a encore correspondance biunivoque entre sous-groupes "réguliers" de Φ et sous-anneaux de R intermédiaires entre R et l'anneau S des invariants de Φ; mais naturellement, les conditions précises caractérisant ces anneaux et les groupes "réguliers" d'automorphismes sont beaucoup plus nombreuses que lorsque R est un anneau simple, et nous devons renvoyer au mémoire lui-même pour leur énoncé précis. Il y a aussi un théorème concernant certains sous-groupes invariants "réguliers" Ψ de Φ, et montrant que Φ/Ψ peut, comme dans la théorie classique, être considéré comme groupe d'automorphismes "régulier" de l'anneau des invariants de Ψ. Enfin, l'auteur prouve un théorème du type classique sur l'extension d'un S-isomorphisme d'un anneau intermédiaire entre S et R, sur un autre anneau de même type: le résultat est qu'un tel isomorphisme peut s'étendre à un automorphisme appartenant à Φ. *J. Dieudonné* (Evanston, Ill.).

Moriya, Mikao. Zur Fortsetzung der 2-Cozyklen in einem kommutativen Ring. Math. J. Okayama Univ. 4, 1–19 (1954).

Let R be a commutative ring with a unit element 1, let R_0 be a subring of R containing 1, and suppose that R has a free R_0-basis of the form $1, \theta, \cdots, \theta^{n-1}$. Let M be an R-module on which 1 acts as the identity operator. The main result is that every symmetric 2-cocycle for R_0 in M can be extended to a symmetric 2-cocycle for R in M. It is shown first that every such cocycle is cohomologous to one which is normalized, in the sense that $f(x, \theta^i)=0$ for all $i=0, 1, \cdots, n-1$ and all $x \in R_0$, and also $f(x\theta, \theta^i)=0$ for $i=0, 1, \cdots, n-2$. The existence of a normalized extension for any given normalized (with respect to 1) 2-cocycle for R_0 in M is then proved by explicit computations extending over 11 pages. Such a normalized extension is determined uniquely by the given cocycle and the arbitrarily prescribable value $f(\theta, \theta^{n-1})$.

It should be pointed out that these results, as well as various generalizations, could easily be obtained without any computations by using the standard interpretation of 2-cohomology in terms of ring extensions. In fact, one starts with the commutative ring extension $(0)\to M\to E\to R_0\to(0)$ which is determined by the given 2-cocycle f for R_0 in M. Then one extends E by adjoining an element y satisfying a suitable defining relation of the form

$$y^n=e_0+e_1y+\cdots+e_{n-1}y^{n-1},$$

so that the homomorphism of E onto R_0 can be extended to a homomorphism of $E[y]$ onto R which sends y into θ. The kernel of this extended homomorphism is $E[y]M$. The R-module structure of M gives rise to a homomorphism of $E[y]M$ onto M whose kernel, K say, is evidently an ideal in $E[y]$. Hence one obtains an extension

$$(0)\to M\to E[y]/K\to R\to(0).$$

Since $E\cap K=(0)$, one may identify E with its canonical image in $E[y]/K$. There is a representative additive isomorphism φ of R_0 into E such that $\varphi(a)\varphi(b)-\varphi(ab)=f(a, b)$, for all $a, b \in R_0$. Extend φ to an additive isomorphism of R into $E[y]$ by defining $\varphi(a\theta^i)$ to be the coset mod K of $\varphi(a)y^i$; $i=0, 1, \cdots, n-1$. Then, if, for $u, v \in R$,

$$f^*(u, v)=\varphi(u)\varphi(v)-\varphi(uv),$$

the map f^* is the required extension of f to a symmetric 2-cocycle for R in M. The normalization properties can easily be read off from the construction. *G. Hochschild.*

Boughon, Pierre. Formule de Taylor pour un polynome à plusieurs indéterminées sur un anneau de caractéristique $p>0$. Application à la détermination des surfaces ayant avec leur plan tangent générique un contact d'ordre >2. C. R. Acad. Sci. Paris 240, 720–722 (1955).

Let A be a commutative ring of characteristic $p>0$; the author determines all polynomials $f \in A[X_1, \cdots, X_n]$ such that in $f(X_1+Y_1, \cdots, X_n+Y_n)$, the terms of total degree 2 in the Y_i vanish. If $p>2$, they are the polynomials $\varphi_0+\sum_{i=1}^n \varphi_iX_i$, where the φ_i belong to $A[X_1^p, \cdots, X_n^p]$; if $p=2$, they have a similar form, the φ_i belonging here to $A[X_1^4, \cdots, X_n^4]$. *J. Dieudonné* (Evanston, Ill.).

Mori, Shinziro. Über die Gleichung $(ax, b)=x$ mit einem unbekannten Ideale x. J. Sci. Hiroshima Univ. Ser. A. 17, 303–309 (1954).

Given two ideals $\mathfrak{a}$, $\mathfrak{b}$ in a Noetherian ring $\mathfrak{o}$, then the ideal $\bigcap_{n=0}^{\infty}(\mathfrak{b}+\mathfrak{a}^n)$ is the largest ideal $\mathfrak{x}$ satisfying $\mathfrak{b}+\mathfrak{a}\mathfrak{x}=\mathfrak{x}$. The structure of this ideal is discussed, without assuming that $\mathfrak{o}$ admits a unit element. *P. Samuel.*

Yoshida, Michio, and Sakuma, Motoyoshi. On integrally closed Noetherian rings. J. Sci. Hiroshima Univ. Ser. A. 17, 311–315 (1954).

It is well known that, if $\mathfrak{o}$ is a Noetherian domain such that, for any proper principal ideal $\mathfrak{o}\mathfrak{a}$ and any associated prime $\mathfrak{p}$ of $\mathfrak{o}\mathfrak{a}$, there is no primary ideal between $\mathfrak{p}$ and $\mathfrak{p}^{(2)}$, then $\mathfrak{o}$ is integrally closed, all the associated primes of $\mathfrak{o}\mathfrak{a}$ are minimal, and every primary ideal belonging to a minimal prime ideal $\mathfrak{p}$ is a symbolic power of $\mathfrak{p}$. These results are extended to Noetherian rings with zero-divisors, with the following modifications: hypotheses and conclusions are restricted to ideals which contain at least a nonzero-divisor, and the total quotient ring of $\mathfrak{o}$ replaces its quotient field. The proof uses Krull's theorem about the intersection of the symbolic powers of a prime ideal, and the method of quotient rings. *P. Samuel* (Cambridge, Mass.).

Yoshida, Michio, and Sakuma, Motoyoshi. The intersection theorem on Noetherian rings. J. Sci. Hiroshima Univ. Ser. A. 17, 317–320 (1954).

The intersection theorem, in its simplest form, asserts that $\bigcap_{n=0}^{\infty}\mathfrak{a}^n=(0)$, $\mathfrak{a}$ being a proper ideal in a Noetherian domain. A more elaborated form studies $\bigcap_{n=\mathfrak{z}}^{\infty}(\mathfrak{b}+\mathfrak{a}^n)$, i.e. the closure of an ideal $\mathfrak{b}$ in the topology defined by the powers of $\mathfrak{a}$. These classical results are extended to the case of a

Noetherian ring $\mathfrak{o}$ without unit element. Two lemmas study the behaviour of prime and primary ideals upon adjunction of a unit element to $\mathfrak{o}$. *P. Samuel* (Cambridge, Mass.).

Lazard, Michel. Bemerkungen zur Theorie der bewerteten Körper und Ringe. Math. Nachr. **12**, 67–73 (1954).

The author gives a short proof of the fundamental theorem on p-adic rings: There exists a unique (up to isomorphism) p-adic ring A with any given perfect residue class ring $\mathfrak{A}$ of characteristic $p<\infty$; if A and A' are p-adic with perfect residue class rings $\mathfrak{A}$ and $\mathfrak{A}'$, every homomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$ is induced by a single homomorphism of A into A'. The proof uses Teichmüller's process [cf. p. 228 of Schilling's Theory of valuations, Math. Surveys, no. 4, Amer. Math. Soc., New York, 1950; MR **13**, 315] for constructing a unique multiplicatively closed residue system R for such a ring A. Let $\alpha \in \mathfrak{A}$, and let $\varphi(\alpha)$ be the representative of α in R. Then each a in A can be represented in the form $\sum_{i=0}^{\infty} \varphi(\alpha_i)p^i$, $\alpha_i \in \mathfrak{A}$. If $*$ represents $+$, $-$, or $\times$, then $\sum_{i=0}^{\infty} \varphi(\alpha_i)p^i * \sum_{i=0}^{\infty} \varphi(\beta_i)$ is again of this form, say $\sum_{i=0}^{\infty} \varphi(\gamma_i)p^i$, where each γ_i is a function of the α_j and β_j, $0 \leq j \leq i$. The present method hinges on a demonstration that these functions are $p^{-\infty}$ polynomial functions. π-adic rings are also treated. *B. N. Moyls* (Vancouver, B. C.).

*Benz, Herbert. **Charakterisierung nichtkommutativer Systeme durch eine bewertungstheoretische Produktformel mit einer Anwendung auf die Einheitentheorie.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 174–178. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

The author defines a pseudovaluation (PV) of a ring R to be a subadditive, submultiplicative function φ on the ring to the reals, such that $\varphi(0)=0$, (1′) $\varphi(a)>0$ if $a\neq 0$, and (2′) for every value x taken on by φ there is an element a_x in R with $\varphi(a_x)=x$ and $\varphi(a_x b)=\varphi(ba_x)=\varphi(a_x)\varphi(b)$ for every $b \in R$. He generalizes a theorem of Artin and Whaples [Bull. Amer. Math. Soc. **51**, 469–492 (1945); MR **7**, 111] as follows. Let R be a simple ring. Then R is an algebra of finite degree over the rational field or over a field of rational functions over a Galois field, if and only if there is an independent set F of PV such that (I) $\prod_{\varphi \in F} \varphi(a)=1$ for all $a\neq 0$ in R, and (II) there is at least one $\varphi \in F$ which is either archimedean or discrete with finite residue class field, and for this φ there is for some $x>1$ an a_x satisfying (2′) and contained in the center. Further result: If R, F satisfy I and II and S is a finite subset of F, then the S-units have a finite basis. This had been proved by C. L. Siegel when the center is a number field [Ann. of Math. (2) **44**, 674–689 (1943); MR **5**, 228].

The PV are a much larger and more loosely organized class of functions than the valuations. They have been studied (without the above assumptions (1′), (2′)) by Cohn and Mahler [Nieuw Arch. Wiskunde (3) **1**, 161–198 (1953); MR **15**, 395]. *G. Whaples* (Bloomington, Ind.).

Munn, W. D. On semigroup algebras. Proc. Cambridge Philos. Soc. **51**, 1–15 (1955).

If S is a finite semigroup and $\mathfrak{F}$ is a field, the algebra $\mathfrak{A}(S)$ of S over $\mathfrak{F}$ is the linear algebra over $\mathfrak{F}$ having the elements of S as a basis. This paper gives several sets of conditions under which $\mathfrak{A}(S)$ is semisimple. Let

$$S=S_1 \supset S_2 \supset S_3 \supset \cdots \supset S_n \supset S_{n+1}=\phi$$

(ϕ the empty set) be a principal series for S, i.e., each S_i is a (two-sided) ideal of S, and there is no ideal of S strictly between S_i and S_{i+1}, for $i=1, 2, \cdots, n$. Then $\mathfrak{A}(S)$ is semisimple if and only if each $\mathfrak{A}(S_i-S_{i+1})$ is semisimple ($i=1, \cdots, n$), where S_i-S_{i+1} denotes the Rees factor semigroup. In particular, this requires that each principal factor S_i-S_{i+1} be simple, in which case S itself is called semisimple. The problem is thus reduced to the case of a simple semigroup S with zero.

Using Rees's representation of such an S as a regular matrix semigroup over a group G with zero [same Proc. **36**, 387–400 (1940); MR **2**, 127], the author shows that $\mathfrak{A}(S)$ can be regarded as the algebra of all $m\times n$ matrices over $\mathfrak{A}(G)$ with multiplication $A \circ B=APB$, where P is the fixed $n\times m$ structure matrix of S. $\mathfrak{A}(S)$ is semisimple if and only if (1) the characteristic of $\mathfrak{F}$ does not divide the order of G, and (2) P is non-singular. Conditions are given for the non-singularity of P in terms of representations of G. A given proper representation of G by matrices in $\mathfrak{F}$ can always be extended to a representation of S, in general in many inequivalent ways. The reviewer has shown [Amer. J. Math. **64**, 327–342 (1942); MR **4**, 4] that there is a unique "basic" extension of smallest possible degree. The author shows that $\mathfrak{A}(S)$ is semisimple if and only if, for every proper representation of G, the basic extension to S is the only one to within equivalence.

For a semigroup S admitting relative inverses, the author shows that $\mathfrak{A}(S)$ is semisimple if and only if (1) the idempotents of S commute, and (2) the characteristic of $\mathfrak{F}$ does not divide the order of the structure group of any principal factor of S. Using the reviewer's description of such a semigroup [Ann. of Math. (2) **42**, 1037–1049 (1941); MR **3**, 199] the author gives an explicit determination of all its representations in terms of those of its structure groups.
A. H. Clifford (Baltimore, Md.).

Büke, Altintas. Untersuchungen über kommutativ-assoziativ und nilpotenten Algebren von Index 3 und von der Charakteristik 2. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19**, supplement, 1–145 (1954).

Verf. untersucht kommutativ-associative Algebren A über einen Körper K der Charakteristik 2, für die $A^3=0$ gilt. Als Strukturideale werden die folgenden Ideale von A bezeichnet: Das Ideal Q aller Elemente deren Quadrate verschwinden; das Ideal R aller x für die $xQ=0$ ist; das Ideal S aller x für die $xR=0$ ist; und verschiedene Ideale die aus Q, R, und S durch die üblichen idealtheoretischen Verknüpfungen hervorgehen. Nachdem der Verf. die Lage der Strukturideale untersucht, behandelt er eingehender die Algebren vom Geschlecht $(n, 1)$ und $(n, 2)$. Dabei heisst A vom Geschlecht (n, m) wenn A/A^2 ein n-dimensionaler Vektorraum und A^2 ein m-dimensionaler Vektorraum über K ist. Die Algebren vom Geschlecht $(n, 1)$ zerfallen dann in 7, die vom Geschlecht $(n, 2)$ in 166 Typenklassen. Der Verf. stellt die Multiplikationstafeln jeder dieser verschiedenen Typen auf und gibt auch die Lage der Strukturideale in allen Fällen sorgfältig an. Die Algebren vom Geschlecht $(n, 1)$ sind schon von Pickert [Math. Ann. **116**, 217–280 (1938)] behandelt worden. *A. Rosenberg.*

Hu, Sze-Tsen. The canonical spaces of associative algebras. Portugal. Math. **13**, 87–96 (1954).

Let A be an associative algebra with unit e over a topologized commutative field F, let M be a two-sided ideal in A, let h_M be the natural homomorphism of A onto A/M. The author calls M "admissible" if there exists an isomorphism k_M of A/M onto F such that $k_M h_M: A \to F$ maps fe onto f for

every $f \in F$. The "canonical" space of A is the space X of all admissible ideals of A, topologized by the usual weak topology (the point-open topology on the space of mappings $k_M h_M$). These ideas are related to work of Hewitt [Trans. Amer. Math. Soc. **64**, 45–99 (1948); MR **10**, 126]. The author's chief interest is in the case that $A=K(x_1, \cdots, x_n)/I$, where I is an ideal in the polynomial ring $K(x_1, \cdots, x_n)$ over the field K of complex numbers; he proves that the canonical space X of A is homeomorphic to the algebraic variety V in n-dimensional affine space K^n consisting of all common zeros of the members of I. Hence the algebraic structure of A and the topology of K determine the topology of V. The author states that this is the first of a series of papers devoted to using this method to compute topological invariants of an algebraic variety V. *S. B. Myers.*

Patterson, E. M. Note on nilpotent and solvable algebras. Proc. Cambridge Philos. Soc. **51**, 37–40 (1955).

Let A be a nonassociative algebra over F. Define

$$A^1=A_1=A^{(0)}=A$$

and

$$A^{i+1}=A^iA, \quad A_{i+1}=AA_i, \quad A^{(i+1)}=A^{(i)}A^{(i)}.$$

Call A nilpotent of class r if r is the smallest integer such that $A^r=A_r=0$ and solvable of index r if $A^{(r)}=0$ but $A^{(r-1)}\neq 0$. It is proved that, if $x^2=0$ for all x in a nilpotent algebra A of dimension $n>1$, then the class of A is at most n. Also, according as the characteristic of F is 0, $p\neq 2$, or $p=2$, the index of any solvable Lie algebra A of dimension $n>2$ is shown to be not greater than $[\log_2(n-2)]+2$, $n-[\frac{1}{3}n]$, or $n-[\frac{1}{4}(n+1)]$. *R. D. Schafer.*

Cohn, Paul Moritz. Sur le critère de Friedrichs pour les commutateurs dans une algèbre associative libre. C. R. Acad. Sci. Paris **239**, 743–745 (1954).

Let A be the free associative algebra of a set E over a field K, let $[A]$ denote the Lie algebra (restricted Lie algebra, if K has characteristic $p>0$) formed by the elements of A relative to commutation, let L denote the subalgebra of $[A]$ generated by E, and let $C_n=L+L^2+\cdots+L^n$. Making use of the well-known Birkhoff-Witt theorem which yields a base of A starting from a base of L, the author gives a very short proof of the criterion, formulated by Friedrichs [Comm. Pure Appl. Math. **6**, 1–72 (1953), esp. footnote 3; MR **15**, 80] and proved by Magnus [Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **BR-3** (1953); MR **15**, 97], for an element f of A to belong to L, and of a new generalization thereof which provides a criterion for f to belong to C_n. *E. R. Kolchin* (Paris).

Frank, Marguerite Straus. A new class of simple Lie algebras. Proc. Nat. Acad. Sci. U. S. A. **40**, 713–719 (1954).

The algebras of the title are subalgebras of the simple Lie algebras of the Witt type and are described as follows: Let $\mathfrak{B}_n$ be that algebra over $\mathfrak{F}$ of characteristic p which has n generators $x_1, \cdots, x_n$ and the Witt algebra $\mathfrak{W}_n$ as derivation algebra. Let $D_1, \cdots, D_n$ be the partial derivations with respect to the corresponding generators, and let $\mathfrak{M}_n=\{D \mid D \in \mathfrak{W}_n \text{ and } \sum_{i=1}^n x_i DD_i=0\}$. $\mathfrak{M}_n$ is a subalgebra of $\mathfrak{W}_n$ and its derived algebra $\mathfrak{M}_n'$ is simple with dimension $(n-1)(p^n-1)$. Other characterizations of $\mathfrak{M}_n'$ are given. *W. G. Lister* (Providence, R. I.).

Hochschild, G. Representations of restricted Lie algebras of characteristic p. Proc. Amer. Math. Soc. **5**, 603–605 (1954).

It is known that all Lie algebras of characteristic p have representations which are not completely reducible. In this note the author proves that this remains true for all restricted Lie algebras except these abelian algebras for which the images $x^{[p]}$, are generators. For any other restricted Lie algebra of n dimensions he constructs explicitly an n-dimensional representation which does not completely split. *W. G. Lister* (Providence, R. I.).

Venkatachaliengar, K., and Srinivasa Rao, K. N. The irreducible representations of the Lie-algebra of the orthogonal group with spin 3/2. Proc. Nat. Inst. Sci. India **20**, 509–523 (1954).

Calcul explicite de certaines représentations linéaires irréductibles des groupes orthogonaux. Ces représentations proviennent de représentations d'une algèbre produit tensoriel d'une algèbre de Clifford et d'une autre algèbre associative définie par un système de générateurs et de relations, grâce auxquelles les auteurs parviennent à la détermination explicite des représentations irréductibles de cette algèbre; ils calculent aussi sa dimension et caractérisent son centre. *J. Dieudonné* (Evanston, Ill.).

Lal, Goverdhan. Construction of a basis for the vector space of remainders in the algebra of Grassmann. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 1046–1057 (1954).

The remainders of any given degree in the Algebra of Grassmann form a vector space of finite dimensionality. We give two methods of constructing a basis of this vector space. Incidentally, we also give a new formula for writing down effectively the remainder of any form. (Author's summary.) *G. Hochschild* (Urbana, Ill.).

Jaeger, Arno. Die Riccatische Differentialgleichung in Körpern der Charakteristik 2. Arch. Math. **5**, 423–428 (1954).

Let K be a field of characteristic $p>0$ in which there is given a system of differentiation in the sense of F. K. Schmidt [J. Reine Angew. Math. **177**, 223–237 (1937)]. F. Kasch showed [Arch. Math. **4**, 17–22 (1953); MR **14**, 841] that if $p>2$ and K contains an element x such that $x^{(1)}=1$, $x^{(n)}=0$ $(n=2, 3, \cdots)$, then the Riccati differential equation $y^{(1)}=ay^2+by+c$ $(a, b, c \in K)$ can always be solved in a quadratic extension of K. The author shows that if $p=2$ the problem breaks up into cases (infinitely many solutions, two solutions, one solution, no solution), the criteria for the various cases being expressed mainly in terms of the vanishing or nonvanishing of $y^{(1)}+by$ at a, at b, and at c. *E. R. Kolchin* (Paris).

Okugawa, Kôtaro. On differential algebra of arbitrary characteristic. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 97–107 (1954).

The Ritt-Raudenbush basis theorem for differential polynomials [see Ritt, Differential algebra, Amer. Math. Soc. Colloq. Publ., v. 33, New York, 1950; MR **12**, 7] was generalized by the reviewer [Trans. Amer. Math. Soc. **52**, 115–127 (1942); MR **3**, 264] to permit as coefficient domain differential rings of a certain kind (a differential field being of this kind precisely when it is perfect). A Seidenberg [ibid. **73**, 174–190 (1952); MR **14**, 130], restricting his attention to coefficient domains which are ordinary differential fields, introduced the concept of "allowable" perfect

differential ideal, and succeeded therewith in proving a reformulated version of the theorem without the hypothesis of perfection. In the present paper the author generalizes Seidenberg's version to include partial differential fields. (Note. In the statement of theorem 2 the adjective "allowable" is inadvertantly omitted.)

Seidenberg's definition of "differentially algebraic" (same reference) is not suitable for partial differential fields; the author adopts another (in the ordinary case, stronger) definition, and shows it leads to a reasonable notion of dimension. However, in seeking (lemma 2) to extend to arbitrary differential fields Seidenberg's result that every nonzero differential polynomial with coefficients in a given ordinary differential field F is not identically zero on F if and only if F is of infinite degree over its field of constants, the author commits an error, and thereby invalidates his proof of the theorem of the existence of a primitive element (theorem 6). In point of fact his lemma 2 and theorem 6 are false, as is shown by the counterexample to theorem 6 in which F is the field of rational functions of a complex variable x, u and v are two analytic functions of x differentially algebraically independent over F, and all elements of $F\langle u, v\rangle$ are considered functions of two complex variables x and y. *E. R. Kolchin* (Paris).

Matsumura, Hideyuki. Automorphism-groups of differential fields and group-varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 283–292 (1954).

The author proves that the Galois group (of strong isomorphisms) of a strongly normal extension of a differential field of characteristic 0 with algebraically closed field of constants C, always has its component of the identity isomorphic in a certain structure-preserving way to a group variety defined over C. This verifies a conjecture of the reviewer [Amer. J. Math. **75**, 753–824 (1953); MR **15**, 394], and makes it possible to carry over to Galois groups certain results on group varieties. *E. R. Kolchin* (Paris).

*Kähler, Erich. **Algebra und Differentialrechnung.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 58–163. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

In spite of the use of a philosophical terminology, this paper contains substantial mathematical results. Part of it is expository. A short dictionary may be useful to readers: unendlich klein = nilpotent; Wirklichkeit = ring in which every non-zero divisor is invertible; Wahrnehmen = non-zero homomorphism; Ursprung eines Wahrnehmen = kernel; Anschauung = factor ring; zentrale Wahrnehmung = homomorphism of a ring A whose kernel is the largest ideal in A (implies that A is a local ring); Erscheinung (einer Wirklichkeit R) = local ring admitting S as total quotient ring; Perspektive = specialization; abgeschlossene Erscheinung = valuation ring; ergründlich = noetherian; Umwelt = completion; K algebraisch über k (für Korpern) = K finitely generated over k.

The first chapter contains generalities about rings and homomorphisms, and also the well known theorem about extensions of specializations, with its applications to integral elements.

In the second chapter one defines the differential algebra $D(A)$ of a commutative ring A: it is the associative algebra generated over A by symbols dx ($x \in A$) submitted to the computation rules $d(x+y) = dx+dy$, $d(xy) = xdy+ydx$, $dx\,dx = 0$. The algebra $D(A)$ is a graded algebra, $D(A) = \sum_{i=0}^{\infty} D^{(i)}(A)$, and is the exterior algebra of the module $D^{(1)}(A)$. The operation d of exterior differentiation is defined in the usual manner, and verifies $d(dx) = 0$. If R is a subring of A, the relative differential algebra $D_R(A)$ is defined as the factor algebra of $D(A)$ by the ideal generated by the elements da ($a \in R$). Some "functor properties" of the operations D, D_R are studied.

A third chapter is devoted to finitely generated field extensions K/k, and to the structure of $D_k(K)$. Relations between the rank of K/k (i.e. the dimension of the vector space $D_k^{(1)}(K)$), its transcendence degree and its separable (or inseparable) character are studied. Separable extensions are defined by the equality between rank and transcendence degree, and the existence of separating transcendence bases is easily derived. The case of inseparable extensions, with the study of p-independence, is discussed in details.

The fourth chapter studies a general notion of differents. Given a ring A and a subring R, denote by $\mathfrak{a}_n$ the ideal in A which is the annihilator of the module $D_R^{(n)}(A) = \Lambda^n D_R^{(1)}(A)$. Assuming that $D_R^{(1)}(A)$ is a finite A-module, we have $\mathfrak{a}_n = A$ for n large enough. Thus the ideals $\mathfrak{d}_j = \prod_{n \geq j} \mathfrak{a}_n$ are defined, and are called the differents of A over R (the author defines these ideals as being generated by some determinants in the coefficients of the linear relations satisfied by the elements of a given basis of $D_R^{(1)}(A)$ over A). The case of field extensions is studied, and also the passage to factor rings, quotient rings and finitely generated ring extensions. The case where A is a discrete-valuation ring, and R the trace of A on some subfield of the quotient field of A is discussed in details, under very general assumptions. The results about the differents $\mathfrak{d}_j$ in this case constitute an ample generalization of the classical relations between differents, ramification indices and inseparability in the residue-field extension.

The last chapter studies essentially the differential forms of a field extension K/k which are "integral" with respect to some set of places of K/k. If one considers the set of all places of K/k, then the differential forms of a given degree of K/k (i.e. the elements of $D_k^{(i)}(K)$) which are integral constitute a finite-dimensional vector space over k. The relations with differential forms of the first kind will be studied in another paper. *P. Samuel.*

Rosenberg, Alex. Finite-dimensional simple subalgebras of the ring of all continuous linear transformations. Math. Z. **61**, 150–159 (1954).

Let $\mathfrak{M}$, $\mathfrak{N}$ be dual vector spaces over a division ring D and let $L(\mathfrak{M}, \mathfrak{N})$ be the ring of all linear transformations T on $\mathfrak{M}$ which possess adjoints with respect to the bilinear functional (x, f) which links $\mathfrak{M}$ and $\mathfrak{N}$. The author studies simple, finite-dimensional subalgebras (over the center of D) of $L(\mathfrak{M}, \mathfrak{N})$ which contain the unit. The centralizer in $L(\mathfrak{M}, \mathfrak{N})$ of such an algebra B is shown to be isomorphic to an $L(\mathfrak{P}, \mathfrak{Q})$, where $\mathfrak{P}$, $\mathfrak{Q}$ are dual spaces over a division ring. (Actually, a somewhat more inclusive result is obtained.) If D is an algebraically closed field Φ, then $\mathfrak{M} = \sum_1^k \oplus \mathfrak{M}_i$, $\mathfrak{N} = \sum_1^k \oplus \mathfrak{N}_i$, $(\mathfrak{M}_i, \mathfrak{N}_j) = 0$ for $i \neq j$, $\mathfrak{M}_i$, $\mathfrak{N}_i$ are dual and the centralizer of B is isomorphic to $L(\mathfrak{M}_i, \mathfrak{N}_i)$, the ring it induces on $\mathfrak{M}_i$. Also $k^2 = [B : \Phi]$. In addition, a characterization is given for those subalgebras of $L(\mathfrak{M}, \mathfrak{N})$ which are centralizers of a simple finite-dimensional subalgebra. In the algebraic closed case, it is also proved that a Φ-isomorphism between two simple finite-dimensional subalgebras can be extended to an inner one of $L(\mathfrak{M}, \mathfrak{N})$ if, and only if, their centralizers are Φ-isomorphic. If $\mathfrak{M}$ is a Hilbert

space, the centralizers are always isomorphic so the isomorphism can always be extended. Finally, some extensions are made to the case of a general division ring D where $\mathfrak{M}$ and $\mathfrak{N}$ have countable dimension over D. In particular every Φ-isomorphism (Φ the center of D) of simple finite-dimensional subalgebras of $L(\mathfrak{M}, \mathfrak{N})$ can be extended to an inner one of $L(\mathfrak{M}, \mathfrak{N})$. *C. E. Rickart* (New Haven, Conn.).

Theory of Groups

Hashimoto, Hiroshi. On a generalization of groups. Proc. Japan Acad. **30**, 548–549 (1954).

The author investigates the structure of a multiplicative system G with elements $a, b, c, \cdots$ and an operator θ which satisfy the three postulates: (1) $(ab)c=a(bc)$, (2) $(a^{\theta}a)b=b$, (3) $(ab)^{\theta}=b^{\theta}a^{\theta}$. It is shown that G is the product of G^{θ}, which is a group, and the subset $C(e)$ consisting of all idempotent elements of G. *D. C. Murdoch.*

Popova, Helen. Logarithmetics of finite quasigroups. I. Proc. Edinburgh Math. Soc. (2) **9**, 74–81 (1954).

Let A be a multiplicative groupoid, not necessarily associative. A power x^r of an element x of A is an element of the subgroupoid generated by x. Here r is not an integer but an index, that is, a symbol used to denote a power. In other words r denotes not only the number of factors x in the power but also the arrangement of brackets. A quasi-integer of A is the class of indices $r, s, \cdots$ such that $x^r=x^s=\cdots$ for all x in A. Addition and multiplication of quasi-integers are defined by $x^{r+s}=x^rx^s$ and $x^{rs}=(x^r)^s$. Multiplication is associative and left distributive with respect to addition but is in general neither commutative nor right distributive. Addition is in general neither commutative nor associative. The set $L(A)$ of all quasi-integers of A with the addition and multiplication thus defined is called the logarithmetic of A. Logarithmetics of quasigroups are investigated and examples are given. The logarithmetic of a finite quasigroup Q is a finite quasigroup with respect to addition. It is also a subdirect union of the logarithmetics of the elements of Q, the logarithmetic of an element being that of the groupoid generated by that element.
D. C. Murdoch (Vancouver, B. C.).

Łos, J. On the existence of linear order in a group. Bull. Acad. Polon. Sci. Cl. III. **2**, 21–23 (1954).

Es wird ein direkter Beweis des bekannten Kriteriums gegeben: Eine Gruppe G ist genau dann total-ordenbar, wenn es für jedes Elementsystem $a_1, \cdots, a_n \neq e$ von G Zahlen $\epsilon_1, \cdots, \epsilon_n$ ($\epsilon_n=\pm 1$) gibt, so dass die aus $a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}$ erzeugte invariante Halbgruppe das Einselement e von G nicht enthält. *P. Lorenzen* (Bonn).

Boone, William W. Certain simple, unsolvable problems of group theory. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. **57**, 231–237, 492–497 (1954); **58**, 252–256 (1955) = Indag. Math. **16**, 231–237, 492–497 (1954); **17**, 252–256 (1955).

These are the first two in a series of papers, with principal result a proof of the recursive unsolvability of a problem closely related to the "word problem" for abstract groups. Magnus [Math. Ann. **106**, 295–307 (1932)] considered the problem of when an element in a group with generators $g_1, \cdots, g_n$ is contained in the subgroup generated by a subset $g_1, \cdots, g_m$ of these generators. The "quasi-Magnus" problem is here shown unsolvable: a group G with generators $g_1, \cdots, g_n$ and a finite set of defining relations is given, for which the problem is recursively unsolvable to decide, for arbitrary $w(g_1, \cdots, g_n)$ representing an element of the group, whether this element is expressible as a product of positive powers of a certain subset $g_1, \cdots, g_m$ of these generators.

The method of proof can be indicated only in roughest outline. A deductive system T_1 is defined, consisting of strings P of symbols Z, together with rules U: $P \rightarrow Q$ for substituting one such string for another; $P \vdash_1 Q$ means that Q is obtainable from P by a succession of such substitutions on its parts. T_1 can be interpreted as describing a certain universal Turing machine M [Turing, Proc. London Math. Soc. (2) **42**, 230–265 (1936)], where the symbols Z stand for the symbols on the tape together with the internal configurations of the machine, and the rules U specify the mode of operation. By a modification of a result of Turing, it is undecidable whether, for a general initial configuration, the machine M ever enters a certain configuration. An argument of Post [J. Symb. Logic **12**, 1–11 (1947); MR **8**, 558] shows this equivalent to the problem whether, for Σ of a certain class, $\Sigma \vdash_1 \Gamma$ for any Γ of a certain other class.

Systems T_2, T_3, T_5 are defined, which are technical elaborations of T_1, and it is shown that the problem $\Sigma \vdash_1 \Gamma$ translates into equivalent problems $\Sigma' \vdash_5 \Gamma'$. In particular, T_5 is obtained by adjoining $Q \rightarrow P$ for every rule $P \rightarrow Q$ of T_3, and the passage from T_3 to T_5 follows an argument of Post [loc. cit.]. Finally, T_6 is obtained by adjoining an inverse $\bar{Z}$ for every symbol Z of T_5, together with rules $Z\bar{Z} \leftrightarrow 1 \leftrightarrow \bar{Z}Z$ (1 the empty string). In short, T_6 is the group G, and the quasi-Magnus problem requires the decision whether $\Sigma \vdash_6 \Gamma$ for some Γ.

It remains to prove the Primary Technical Result: that $\Sigma \vdash_5 \Gamma$ for some Γ if and only if $\Sigma \vdash_6 \Gamma'$ for some Γ'. This is the heart of the argument, and its proof, by an extension of methods of Malcev [Mat. Sb. N.S. **6(48)**, 331–336 (1939); **8(50)**, 251–264 (1940); MR **2**, 7, 128], is begun in the papers under review. *R. C. Lyndon* (Ann Arbor, Mich.).

Eilenberg, Samuel, and MacLane, Saunders. On the homology theory of abelian groups. Canad. J. Math. **7**, 43–53 (1955).

The authors have previously introduced the notion of generically acyclic constructions [Trans. Amer. Math. Soc. **71**, 294–330 (1951); MR **13**, 314] to yield a homology theory of abelian groups different from the familiar one based on (freely) acyclic chain-complexes. In their earlier paper they proved the existence of generically acyclic constructions for arbitrary classes of multiplicative systems (with unit element) defined by identities, and the uniqueness of the resulting homology theory, and also exhibited an explicit (cubical) construction for the class of associative and multiplicative systems. A second construction, $A(\Pi)$, was suggested, based on the freely acyclic construction $A^*(\Pi)$, and in this paper this construction is described completely.

It turns out that the construction $A(\Pi)$ is intimately related to that of the complexes $A(\Pi, n)$, introduced by the authors to calculate the groups $H(\Pi, n)$ [Ann. of Math. (2) **58**, 55–106 (1953); MR **15**, 54]. The definition of $A(\Pi, n)$ is modified slightly so that the suspension $A(\Pi, n) \rightarrow A(\Pi, n+1)$ may be regarded as an inclusion (of additive graded groups with boundary) and $A(\pi)$ is then the union of the (modified) $A(\pi, n)$; so defined, $A(\Pi)$ starts with $J(\Pi)$ in dimension 1 and is augmented by adding a suitable 0-dimensional component and boundary operator from $J(\Pi)$ to become

acyclic in dimensions <2. It turns out that in the resulting homology theory (which must, by the authors' previous result, be isomorphic with that given by the cubical construction), $H_q(A(\Pi))$ is the stable group $H_{q-1+n}(\Pi, n)$, $q<n+1$.

The proof of the generic acyclicity of $A(\Pi)$ requires the observation that the generic elements of $A(\Pi, n)$ span a subcomplex of $A(\Pi, n)$; this result the authors establish by an analysis of homomorphisms of tensor products of considerable interest in itself.

Let R be an augmented commutative ring (with augmentation α) and let R^m be its m-fold tensor product; consider the functors (of t variables) $R_1^{m_1}\otimes\cdots\otimes R_t^{m_t}$. Then $\rho\colon R_1^{m_1}\otimes\cdots\otimes R_t^{m_t}\to R_1^{n_1}\otimes\cdots\otimes R_t^{n_t}$ is a natural transformation (of functors) if it commutes with the homomorphisms induced by augmented ring homomorphisms $\beta_k\colon R_k\to Q_k$, $k=1, \cdots, t$. The authors prove that the set of all natural group homomorphisms ($m_1, \cdots, m_t$; $n_1, \cdots, n_t$ being held fixed) forms a free abelian group with the set of all natural ring homomorphisms as basis. A natural ring-homomorphism may be written uniquely as $\rho=\rho_1\otimes\cdots\otimes\rho_t$, and each such ρ_k is a composition of natural augmented ring homomorphisms of the following four types:

(i) $\rho\colon R^m\to R^{m+1}$, given by
$$x_1\otimes\cdots\otimes x_m\to x_1\otimes\cdots\otimes x_m\otimes 1,$$

(ii) $\rho\colon R^{m+1}\to R^m$, given by
$$x_1\otimes\cdots\otimes x_{m+1}\to(\alpha x_1)x_2\otimes x_3\otimes\cdots\otimes x_{m+1},$$

(iii) $\rho\colon R^{m+1}\to R^m$, given by
$$x_1\otimes\cdots\otimes x_{m+1}\to x_1x_2\otimes x_3\otimes\cdots\otimes x_{m+1},$$

(iv) $\rho\colon R^m\to R^m$, given by
$$x_1\otimes\cdots\otimes x_m\to x_{\sigma 1}\otimes\cdots\otimes x_{\sigma m},$$

where σ is an arbitrary permutation of $1, \cdots, m$. The proof of this theorem uses the polynomial ring $J[y_1, \cdots, y_m]$ in the indeterminates $y_1, \cdots, y_m$, as "universal example".

[On p. 47, l. 2 from bottom, "$\kappa_j y_j$" should be "$\kappa_j y_i$"; on p. 48, l. 2, insert "ring-homomorphism" after "some"; on p. 48, l. 7 and l. 9, replace "$i \in A$" and "$x_i \in A_j$" respectively by "$i \in A_j$".] *P. J. Hilton.*

Sąsiada, E. On abelian groups every countable subgroup of which is an endomorphic image. Bull. Acad. Polon. Sci. Cl. III. **2**, 359–362 (1954).

Kertész and Szele characterized abelian groups every finitely generated subgroup of which is an endomorphic image [Acta Sci. Math. Szeged **15**, 70–76 (1953); MR **15**, 196]. The author shows that an abelian group has (P). Every countable subgroup is an endomorphic image, if and only if the group possesses a direct summand which is the direct sum of an infinite number of copies of the group of integers or if it be the direct sum of a torsion group with (P) and a finite (possibly zero) number of copies of the group of integers. A torsion abelian group has (P) if and only if every p-primary component which has no universal bound on the orders of its elements have a homomorphic image which is the direct sum of all the subgroups of the group of type (p^∞). Thus, if an abelian group is countable, every subgroup thereof is an endomorphic image if and only if the group splits as above. A torsion-free abelian group has every subgroup an endomorphic image if and only if it be either finitely generated or have a direct summand which is the direct sum of copies of the group of integers, the number of copies having the same power as the group itself.

F. Haimo (St. Louis, Mo.).

Green, J. A. On the converse to a theorem of R. Brauer. Proc. Cambridge Philos. Soc. **51**, 237–239 (1955).

Let G be a group of finite order and let $\mathfrak{E}$ denote the set of all elementary subgroups E, i.e. of all subgroups which are a direct product of a cyclic subgroup and a p-group for some prime p. The reviewer has proved the following two equivalent statements. (A) A class-function θ on G is a generalized character of G, if and only if the restriction of θ to E is a generalized character of E for each $E \in \mathfrak{E}$. (B) Each character of G is a linear combination with integral rational coefficients of characters induced by characters of the subgroups $E \in \mathfrak{E}$ [see R. Brauer, Ann. of Math. (2) **48**, 502–514 (1947); **57**, 357–377 (1953); MR **8**, 503; **14**, 844]. It is clear that these statements remain true if $\mathfrak{E}$ is replaced by a set of subgroups such that each elementary subgroup E of G is conjugate to a subgroup of a group in $\mathfrak{E}$. The author proves the converse: If $\mathfrak{E}$ is a set of subgroups of G, and if either of the above statements is true, then every elementary subgroup E of G is conjugate to a subgroup of a group of $\mathfrak{E}$.

R. Brauer (Cambridge, Mass.).

Černikov, S. N. Groups with systems of complemented subgroups. Mat. Sb. N.S. **35(77)**, 93–128 (1954). (Russian)

This paper contains the proofs of the results announced in a note by the same author [Dokl. Akad. Nauk SSSR (N.S.) **92**, 891–894 (1953); MR **15**, 504; see also Baeva, ibid. **92**, 877–880 (1953); MR **15**, 503]. There follow some results, not mentioned in the reviews of these notes. Every abelian subgroup of a locally finite group G is complemented in G if and only if G splits into a semi-direct product of an abelian normal subgroup A and an abelian subgroup B such that every subgroup of A and every subgroup of B is complemented in G. If A is a normal abelian subgroup of a group G and if every serving subgroup of A is complemented in G, then A is complemented in G, every serving subgroup of A is complemented in A and A is the direct sum of rational groups, subgroups of the rational group, groups of type (p^∞) and subgroups of groups of type (p^∞) such that each summand is normal in G. Every serving subgroup of an abelian group A is complemented in A if and only if A is a direct sum S of the type mentioned just above where (1) there do not exist two non-isomorphic, non-complete torsion-free summands of S, where (2) there does not exist an infinite sequence of torsion-free, non-complete summands of S and where (3) there does not exist among the summands of S an infinite sequence of groups of orders p^{n_i} with $\{n_i\}$ unbounded increasing. *F. Haimo* (St. Louis, Mo.).

Higman, D. G. Remarks on splitting extensions. Pacific J. Math. **4**, 545–555 (1954).

On the basis of theorems of Gaschütz [J. Reine Angew. Math. **190**, 93–107 (1952); MR **14**, 445] the author proves, among other things, the following. If N is a normal subgroup of the finite group G, then the subgroup C is a complement of N if and only if it is minimal with respect to the property $G=NC$, and for each prime p there is a Sylow p-subgroup S of G such that $S\cap N$ has a complement in S which is part of C. Schur's theorem, on the complementation of a normal subgroup N whose order and index are coprime, is a corollary. The author also obtains a condition that a normal p-subgroup should be contained in the Frattini subgroup $\Phi(G)$; and observes that if A is a complemented abelian normal subgroup of G, then any two conjugates of A in G are conjugate if, for each prime p, there is a Sylow p-subgroup S of G such that any two complements of $S\cap A$

in S are conjugate in S. The conjecture that in Schur's theorem above any two complements of N are necessarily conjugate is shewn to be equivalent to the proposition that a group of automorphisms of the finite group G, of order prime to the order of G, must leave some Sylow p-subgroup of G invariant, for each prime p. The paper concludes with an example, due to Zassenhaus, of a finite group G with a (necessarily non-abelian) normal p-subgroup which is complemented in a Sylow p-subgroup of G, but not in G itself. *Graham Higman* (Oxford).

Petresco, Julian. Sur les commutateurs. Math. Z. **61**, 348–356 (1954).

Let $A^1, A^2, \cdots, A^\nu$ be subgroups of any group and let $O^k = O^k(A^1, \cdots, A^\nu)$, where $0 \leqq k < \nu$, be the join of all subgroups formed by commutation of the A^i and which involve more than k distinct A^i's. The author characterises the elements of O^k in the following way. Every such element can be expressed as a product $\prod_i a_i^1 a_i^2 \cdots a_i^\nu$, where each $a_i^\alpha \in A^\alpha$, in such a way that the result of suppressing in this product all those factors a_i^α for which α takes one of λ assigned values, where $\lambda \geqq \nu - k$, is always equal to 1, however the λ values are chosen. Conversely, every element having an expression with this property belongs to O^k. It is shown further that the group generated by the A^i is

$$O^0 = \prod_{k_1} A^{k_1} \cdot \prod_{k_1, k_2} O^1(A^{k_1}, A^{k_2}) \cdots \prod_{k_1, k_2, \cdots, k_\lambda} O^{\lambda-1}(A^{k_1}, \cdots, A^{k_\lambda}) \cdots O^{\nu-1};$$

and that, if a, b are elements of a group and if the elements $c_2, c_3, \cdots$ are defined by recurrence from the formulae $(ab)^\lambda = a^\lambda b^\lambda c_2^{\binom{\lambda}{2}} c_3^{\binom{\lambda}{3}} \cdots c_{\lambda-1}^{\binom{\lambda}{\lambda-1}} c_\lambda$ $(\lambda = 2, 3, \cdots)$, then $c_\lambda \in O^{\lambda-1}\{a, b\}$, the λth term of the lower central series of the group generated by a and b. This is the canonical form given to a formula of the reviewer's by M. Lazard [C. R. Acad. Sci. Paris **236**, 36–38 (1953); MR **14**, 617] using a less elementary method. The paper contains several other results on commutator subgroups. *P. Hall.*

Keller, Ott-Heinrich. Eine Darstellung der Komposition endlicher Gruppen durch Streckenkomplexe. Math. Ann. **128**, 177–199 (1954).

This paper investigates groups G with finite composition series. The main interest is in the various composition series of G and the order of the composition factors in them. He observes that in the direct product of two simple groups A and B, the groups A and B permute elementwise as direct factors, and also as to their order in a composition series. Various permutability properties of this type are considered. He defines a pseudo-simple group as either a simple group or a p-group, and in terms of these a pseudo-composition series. A group is called loose if every pseudo-composition series is a composition series. He shows that the factors of a loose group retain their identity in that the interchanges of factors obtained by going from one composition series to another will reduce to the identity when we return to the original chain. This is in distinction to what can happen if the composition groups form a lattice which is modular but not distributive. *Marshall Hall, Jr.* (Columbus, Ohio).

Osima, Masaru. Some remarks on the characters of the symmetric group. II. Canad. J. Math. **6**, 511–521 (1954).

In a previous paper [same J. **5**, 336–343 (1953); MR **15**, 100], the author has determined the number of irreducible modular representations in a given p-block of the symmetric group. In the present paper, a simpler proof is given. Properties of the decomposition numbers are obtained for the case of the symmetric group, and it is shown that the elementary divisors of the matrix of the Cartan invariants of a block are determined by the prime p and the weight of the block. *R. Brauer* (Cambridge, Mass.).

Littlewood, D. E. On orthogonal and symplectic group characters. J. London Math. Soc. **30**, 121–122 (1955).

Using properties of Schur functions, the author makes explicit two formulae of Ibrahim [Proc. Nat. Acad. Sci. U. S. A. **40**, 306–309 (1954); MR **15**, 853] connecting symplectic and orthogonal group characters. *G. de B. Robinson* (Toronto, Ont.).

Waelbroeck, L. Le groupe symplectique et ses représentations. Bull. Soc. Math. Belgique **6** (1953), 47–52 (1954).

Outline (without details) of a purely algebraic proof of the complete reductibility of all tensor spaces under the symplectic group (over an arbitrary field of characteristic 0). The main ideas consist, on one hand, of decomposing each symplectic $2n \times 2n$ matrix in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I & Z' \\ 0 & I \end{pmatrix}$$

and considering the subgroups consisting of matrices having only one of the three components distinct from the identity; on the other hand, the author selects a priori certain types of tensors (related to the fundamental covariant tensor representing the bilinear alternate form invariant under the group), and considers how the three particular subgroups considered above act on these tensors. *J. Dieudonné.*

Goto, Ken-iti. Wave fields in de Sitter space. Progr. Theoret. Phys. **12**, 311–354 (1954).

The four-component spinor representation of the Lorentz group in five dimensions is discussed in detail except for the discussion of the reality conditions. This theory is then applied to the discussion of general wave equations for particles with arbitrary spin in de Sitter space. Solutions of the second-order wave equations which have to be satisfied by each component of the spinor field are given. The references to the literature on this subject are not complete. *A. H. Taub* (Urbana, Ill.).

Gotô, Ken-iti. Relativistic wave equations under the inhomogeneous Lorentz group. Progr. Theoret. Phys. **12**, 409–420 (1954).

The author uses the fact that the inhomogeneous Lorentz group is a subgroup of the conformal group in four dimensions and the additional fact that the latter group is isomorphic to the group of the quadratic form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2$$

to obtain an eight-component spinor representation of the inhomogeneous Lorentz group. He also proposes wave equations to be satisfied by such eight-component spinors with various numbers of indices. *A. H. Taub.*

Naĭmark, M. A. Linear representations of the Lorentz group. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 19–93 (1954). (Russian)

An exposition in relatively elementary terms of the known irreducible linear representations of the proper Lorentz

group, including the finite-dimensional and all the continuous unitary such representations. *I. E. Segal.*

Naĭmark, M. A. On the description of all unitary representations of the complex classical groups. I. Mat. Sb. N.S. **35(77)**, 317–356 (1954). (Russian)

This commences the detailed exposition of a previously announced result [Dokl. Akad. Nauk SSSR (N.S.) **84**, 883–886 (1952); MR **14**, 16]. The proof shows directly that any continuous unitary irreducible representation of one of the classical complex simple Lie groups is a direct integral of the known irreducible unitary representations given by Gelfand and Naĭmark [Trudy Mat. Inst. Steklov. **36** (1950); MR **13**, 722], which implies the main result that no other irreducible unitary representations exist. The proof is based on the reduction of a positive linear functional on the group ring to a positive linear functional on an abelian ring related to the highest weights of representations of the unitary subgroup, and subsequent examination of the functionals on this abelian ring. While this is formally somewhat parallel to the procedure used by Gelfand and Naĭmark [Izv. Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); MR **9**, 495] to settle the same question for the case of the Lorentz group, the present approach is essentially different from the earlier one, the abelian ring involved being distinct. The characterization of its elements is one of the main steps in the proof. *I. E. Segal* (Chicago, Ill.).

Graev, M. I. Principal series of unitary representations of real forms of the complex unimodular group. Dokl. Akad. Nauk SSSR (N.S.) **98**, 517–520 (1954). (Russian)

All real forms of a complex simple Lie group are known to arise from automorphisms of period 2 of the compact form of the group. This note describes the principal series of irreducible unitary representations for those real forms of the complex unimodular group that arise from inner automorphisms. The representations are given in global form, the basic series being in a space of functions of several complex variables. The real unimodular groups, which constitute one of the main cases when the associated automorphism is outer, have been treated by Gelfand and Graev [Izv. Akad. Nauk SSSR. Ser. Mat. **17**, 189–248 (1953); MR **15**, 199]. The present groups are the subgroups of the complex unimodular groups leaving invariant certain non-degenerate hermitian forms. *I. E. Segal* (Chicago, Ill.).

Kim, Sen En. On imprimitive Lie groups. Dokl. Akad. Nauk SSSR (N.S.) **99**, 205–207 (1954). (Russian)

The author treats the classification of imprimitive groups of transformations. Although solved in a sense by Morosoff and Dynkin, certain questions remain. The author gives a method of constructing the infinitesimal operators of an imprimitive group and applies it to the redetermination of the imprimitive groups in the plane, and to the determination of the nonsolvable groups in 3-space. In the latter case the possible solvable groups are of such large dimension that the present method is impractical. *I. E. Segal.*

Bernard, Daniel. Sur la structure des pseudogroupes de Lie. C. R. Acad. Sci. Paris **239**, 1263–1265 (1954).

Let V_n be an n-dimensional differentiable manifold with a G-structure S [S. S. Chern, Géométrie différentielle, Colloq. Internat. du Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 119–136; MR **16**, 112] given by an open covering $\{U_\alpha\}$ and $n\times 1$ matrices θ_α of Pfaffian forms defined in U_α. For any normalizator h of G in the general real linear group $GL(n, R)$ with h non-ε G, the system $\{U_\alpha, h\theta_\alpha\}$ gives a new G-structure S' over V_n which will be called a G-structure associated with S. The following two theorems are proved. (1) In order that two isotropic (in the sense of Libermann) and homogeneous G-structures admit the same automorphisms, it is necessary and sufficient that they are associated. (2) Suppose G to be involutive and S an integrable, real analytical G-structure. Then the G-structures associated with S are all integrable. Moreover, they are the only G-structures which admit the same automorphisms as S.

By a normal transitive pseudo-group of Lie, the author means the pseudo-group of automorphisms of an involutive, integrable and real analytical G-structure. Certain invariants, called reduced structure constants, are introduced. The pseudo-groups corresponding to associated G-structures have the same reduced structure constants. The author states that two normal transitive pseudo-groups of Lie are locally similar if and only if they have the same reduced structure constants. *H. C. Wang* (Princeton, N. J.).

Raševskiĭ, P. K. Inner algebraic Lie groups. Dokl. Akad. Nauk SSSR (N.S.) **98**, 539–540 (1954). (Russian)

A Lie group is said to be inner algebraic if its adjoint group is algebraic. The following results are stated without proof: A complex Lie group G is algebraic if and only if one can find in its Lie algebra such a basis b_α, a_α, s_ω, that the a_α, b_α generate the radical R of the algebra, the b_α generate a nilpotent ideal of R, the s_ω generate a semi-simple subalgebra, and the relations $[a_\alpha, a_\beta]=0$, $[a_\alpha, b_\beta]=\mu_{\alpha\beta}b_\alpha$, $[a_\alpha, s_\omega]=0$ hold, where the $\mu_{\alpha\beta}$ are integers. The same conditions are also necessary and sufficient for G to have a (locally faithful ?; faithful ?) algebraic representation (it seems to the reviewer that certain hypotheses are implicitly made here, such as perhaps the simple connectedness of G); when this is the case, every unimodular representation of G which is not the direct sum of two representations can be changed into an algebraic representation by adding suitable nilpotent matrices to the matrices corresponding to the a_α, and leaving unchanged the matrices corresponding to the b_α and the s_ω, in the given representation. *J. L. Tits.*

Dobrescu, Andrei. La classification des groupes de Lie réels à quatre paramètres. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 395–436 (1953). (Romanian. Russian and French summaries)

The purpose of this paper is the classification of all 4-dimensional real Lie algebras; this has been done for the complex 4-dimensional algebras by S. Lie [Theorie des Transformationsgruppen, Bd III, Teubner, Leipzig, 1930], and for the real 3-dimensional algebras by L. Bianchi [Lezioni sulla teoria dei gruppi continui finiti di trasformazioni, Spoerri, Pisa, 1918], G. Vranceanu [Leçons de géométrie différentielle, v. I, Rotativa, Bucarest, 1947; MR **9**, 532] and H. C. Lee [J. Math. Pures Appl. (9) **26**, 251–267 (1948); MR **9**, 567]. The author reduces all possible 4-dimensional structure tensors c^h_{kl} to 13 explicitly given canonical forms; the proofs consist for a large part in computations involving mainly, aside from c^h_{kl}, the tensors $c_k=c^a_{ak}$ and $c_{kl}=c^a_{kb}c^b_{la}$, already used by Vranceanu [loc. cit.]. A few general (i.e. valid for all dimension) lemmas and identities are proved. In the last part of the paper (chapter IV), the author checks that, going from the reals to the complexes, his results reduce to Lie's. *J. L. Tits.*

Tits, Jacques. Étude géométrique d'une classe d'espaces homogènes. C. R. Acad. Sci. Paris **239**, 466–468 (1954).

Let G be a complex semi-simple Lie group, $\bar{G}$ its complex Lie algebra, $\bar{C}$ a Cartan subalgebra of $\bar{G}$, and

$$\Sigma = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$$

a system of simple roots (fundamental angular parameters). For a subset Π of Σ, denote by $\bar{G}(\Pi)$ the subalgebra spanned by $\bar{C}$ and all the root vectors e_α corresponding to the roots $\alpha = \sum_i^r k_i\alpha_i$ (k_i = integers) such that $k_i \geqq 0$ whenever $\alpha_i \in \Pi$. Let $G(\Pi)$ denote the subgroup generated by $\bar{G}(\Pi)$. Then the coset space $G[\Pi] = G/G(\Pi)$ is called an R-space.

The author associates, to each R-space $G[\Pi]$ a diagram which is obtained from the Schläfli figure of Σ by marking distinguishably those vertices representing roots in Π. It is stated that two R-spaces are isomorphic if and only if their diagrams differ only by a connected part in which no vertex is marked. [Reviewer's remark: The class of R-spaces coincides with the class of compact, homogeneous complex manifolds with non-vanishing Euler characteristic. Thus the author's diagram gives a simple way to write out such spaces.]

Let Π, Π' be two subsets of Σ. A point of $G[\Pi]$ and a point of $G[\Pi']$ are called incident if they are represented, respectively, by the cosets $gG(\Pi)$, $gG(\Pi')$ for a same $g \in G$. A Π'-plane in $G[\Pi]$ is defined to be the totality of points of $G[\Pi]$ incident with a fixed point of $G[\Pi']$. Properties of Π'-planes in $G[\Pi]$, especially the inclusion and intersection relations of Π'-planes with different Π', are studied.

In the last paragraph, the author uses the Π'-planes to discuss the geometry on an R-space of 16 complex dimensions belonging to the exceptional group E_8 (Cartan's symmetric space of type EIII). *H. C. Wang.*

Tits, Jacques. Sur les R-espaces. C. R. Acad. Sci. Paris **239**, 850–852 (1954).

This note concerns the imbedding, in the complex projective space P, of an R-space $G[\Pi]$ defined in the paper reviewed above. Let f: $G[\Pi] \to P$ be an analytical homeomorphism of G into P, and let V denote the image of $G[\Pi]$ under f. Suppose that, for each g of G, the automorphism fgf^{-1} of V can be extended to a projectivity γ of P, and that V is not contained in any proper linear subspace of P. Then γ is uniquely determined by g, and the mapping f_0: $g \to \gamma$ gives a projective representation of the group G. The main theorem reads as follows: "Let φ be an analytical homomorphism of a complex semi-simple Lie group G in the group of all projectivities of a complex projective space P. Then φ is irreducible if and only if there exist an R-space $G[\Pi]$ of G and a mapping f: $G[\Pi] \to P$ having all the above properties such that $\varphi = f_G$. If this is the case, then both f and Π are determined by φ, and moreover, Π consists of all the simple roots α_i such that $(\rho, \alpha_i) \neq 0$, where ρ is the dominant weight of φ, and (x, y) denotes the Cartan's bilinear form of G." This has some relations with the paper of M. Goto reviewed below. *H. C. Wang.*

Goto, Morikuni. On algebraic homogeneous spaces. Amer. J. Math. **76**, 811–818 (1954).

A C-space is here a compact, complex, complex-homogeneous manifold with a finite fundamental group [for the classification of C-spaces see H. C. Wang, same J. **76**, 1–32 (1954); MR **16**, 518]; it is called semisimple if its Euler characteristic is not zero. The principal result is that every semisimple C-space V is an algebraic and even a rational variety (i.e., birationally equivalent to complex projective space). More precisely, if $V = K/U$, where K is a compact Lie group of dim r and U a subgroup of maximal rank, and dim $V = n$, then V can be imbedded in the Grassmann manifold $G_{r,n}$ of n-spaces in r-space (complex). Also, V is a direct product of simple C-spaces (i.e. with K simple) and it is simply connected; any C-space is a complex-analytic fiber bundle with a torus as fiber and a rational variety for base. If $H(r)$ denotes the subgroup of $GL(r, C)$, which maps the space spanned by the first n axes into itself, then one gets an imbedding of $GL(r, C)/H(r)$ into $G_{r,n}$ by mapping each matrix into the Grassmann product of the first n columns. If $G \subset GL(r, C)$ is the adjoint group of the group of complex automorphisms of V, then it is proved by a detailed analysis that the stability group H can be taken as $G \cap H(r)$; the above imbedding of $GL(r, C)/H(r)$ induces now an imbedding of V, and it is shown that the image is actually rational. An important role is played by the unique decomposition of $GL(r, C)$ (up to "singular" elements, i.e., matrices for which one of upper-left hand principal minors is singular) into the product $N(r) \times P(r)$, where $N(r)$ consists of the matrices with 0 above and 1 on the diagonal, and $P(r)$ consists of the matrices with 0 below the diagonal; the group G has a similar decomposition corresponding to all negative roots (for N) and all positive roots plus the Cartan subalgebra (for P). *H. Samelson.*

Leptin, Horst. Bemerkung zu einem Satz von S. Kaplan. Arch. Math. **6**, 139–144 (1955).

Let $\mathfrak{Q} = \{G_\mu\}_{\mu \in I}$ be a system of abelian topological groups indexed by a set I and let $\Theta = \{H_\mu\}_{\mu \in I}$ be a system of closed subgroups $H_\mu \subset G_\mu$. The author defines the Θ-direct sum $G^\Theta = \sum_{\mu \in I}^\Theta G_\mu$ of the system $\mathfrak{Q}$ to be the set of all systems of elements $\{x_\mu\}_{\mu \in I}$ such that $x_\mu \in H_\mu$ for almost all values of μ in I. A suitable topology of G^Θ is defined by choosing a fundamental system of neighborhoods of the null element. With this definition, the author proves a generalization of a theorem of S. Kaplan [Duke Math. J. **15**, 649–658 (1948); MR **10**, 233] stated as follows: Let $\mathfrak{Q} = \{G_\mu\}_{\mu \in I}$ and $\hat{\mathfrak{Q}} = \{\hat{G}_\mu\}_{\mu \in I}$ be two systems of abelian topological groups indexed by the same set I such that, for each $\mu \in I$, G_μ and $\hat{G}_\mu$ are dual groups (i.e. each is the character group of the other). Suppose that, to each $\mu \in I$, there has been given a subgroup $H_\mu \subset G_\mu$ and a subgroup $\hat{H}_\mu \subset \hat{G}_\mu$ such that each is the annihilator of the other. Let $\Theta = \{H_\mu\}_{\mu \in I}$ and $\hat{\Theta} = \{\hat{H}_\mu\}_{\mu \in I}$. Then G^Θ and $\hat{G}^{\hat{\Theta}}$ are dual groups; furthermore, the subgroups $\sum_{\mu \in I} H_\mu \subset G^\Theta$ and $\sum_{\mu \in I} \hat{H}_\mu \subset \hat{G}^{\hat{\Theta}}$ are annihilators of each other. If, for each $\mu \in I$, $(H_\mu, \hat{G}_\mu/\hat{H}_\mu)$ and $(\hat{H}_\mu, G_\mu/H_\mu)$ are pairs of dual groups, then so are the pairs $(\hat{G}^{\hat{\Theta}}/\sum_\mu \hat{H}_\mu, \sum_\mu H_\mu)$ and $(G^\Theta/\sum_\mu H_\mu, \sum_\mu \hat{H}_\mu)$. If $\Theta = \mathfrak{Q}$ and $\hat{\Theta} = 0$, then this theorem reduces to that of S. Kaplan. *S. T. Hu* (New Orleans, La.).

Corduneanu, C. Sur un théorème de Sze-Tsen Hu. Acad. Repub. Pop. Romîne. Fil. Iași. Stud. Cerc. Ști. **5**, 45–47 (1954). (Romanian. Russian and French summaries)

The (demonstrated) fact that in some connected topologic groups any bounded open set is void is said to conflict with a theorem (IV) of Hu's [Portugal. Math. **6**, 49–56 (1947); MR **9**, 297]. *R. Arens* (Los Angeles, Calif.).

Wallace, A. D. Topological invariance of ideals in mobs. Proc. Amer. Math. Soc. **5**, 866–868 (1954).

A mob is a (Hausdorff) topological semigroup. A clan is a compact, connected mob with unit. It is known [Numa-

kura, Math. J. Okayama Univ. 1, 99–108 (1952); MR 14, 18] that every compact mob contains a unique minimal (two-sided) ideal which is closed. H. Cohen [Duke Math. J. 21, 209–224 (1954); MR 16, 609] has defined the codimension cd (X, G) of a locally compact space X with respect to a coefficient group G to not exceed n if the natural homomorphism of (the Alexander-Kolmogoroff cohomology group) $H^*(X, G)$ into $H^*(A, G)$ is onto, for every closed subset A of X. He has shown that for compact X, cd (X, G) does not exceed the ordinary covering dimension, and coincides with it if G is the integers.

The main result of the paper reviewed is: if S is a clan such that cd $(S, G) \leq n$ for some group G such that $H^n(S, G) \neq 0$, then any homeomorphism of S upon a clan T sends the minimal ideal of S upon the minimal ideal of T.

M. Henriksen (Lafayette, Ind.).

NUMBER THEORY

*Hasse, G. [H.] **Lekcii po teorii čisel.** [Lectures on the theory of numbers.] Izdat. Inostrannoĭ Lit., Moscow, 1953. 527 pp. 25.40 rubles.

Translation of the author's Vorlesungen über Zahlentheorie [Springer, Berlin, 1950; MR 14, 534].

Thébault, Victor. Suites de carrés parfaits remarquables. Mathesis 63, nos. 9–10, supplément, 1–8 (1954).

Thébault, Victor. Sur les nombres qui terminent les carrés parfaits. Mathesis 63, nos. 9–10, supplément, 8–11 (1954).

Iyer, R. V., et Delcourte, M. Nombres triangulaires remarquables. Mathesis 63, nos. 9–10, supplément, 11–15 (1954).

Thébault, Victor, et Iyer, R. V. Nombres triangulaires spéciaux. Mathesis 63, nos. 9–10, supplément, 16–23 (1954).

Bang, Thøger. Large prime numbers. Nordisk Mat. Tidskr. 2, 157–168, 191 (1954). (Danish. English summary)

This is an historical survey of the numbers $M_n = 2^n - 1$ of Mersenne beginning with Euclid, through the renaissance and the eras of Fermat-Mersenne, Euler, and Lucas to the present day results of the SWAC. The test of Lucas for the primality of $2^n - 1$ is proved by a paraphrase of Lucas' functions via trigonometric functions of complex argument (in reality hyperbolic functions). The largest known prime $2^{2281} - 1$ is written out as a 687 digit decimal integer.

D. H. Lehmer (Berkeley, Calif.).

Halberstam, H. On the distribution of additive number-theoretic functions. J. London Math. Soc. 30, 43–53 (1955).

Let $f(n)$ be an additive function. Put

$$A_n = \sum_{p<n} \frac{f(p)}{p}, \quad B_n = \sum_{p<n} \frac{(f(p))^2}{p}.$$

Assume that $|f(p)| < c$ and that $B_n \to \infty$ as $n \to \infty$. The author proves that for every positive integer k

$$(1) \quad \lim_{n=\infty} \sum_{m=1}^{n} (f(m) - A_n)^k / n B_n^{k/2} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} x^k e^{-x^2/2} dx.$$

The proof is elementary but fairly complicated. The author then deduces from (1) the theorem of Erdös and Kac [Amer. J. Math. 62, 738–742 (1940); MR 2, 42] without using the central-limit theorem and Brun's method, this fulfills a desideratum of Kac [Bull. Amer. Math. Soc. 55, 641–665 (1949); MR 11, 161].

P. Erdös.

Kanold, Hans-Joachim. Über die Dichte von gewissen Zahlenmengen. J. Reine Angew. Math. 193, 250–252 (1954).

Let $a, d, \cdots$ denote positive integers. For every a let $(Q(a))^2$ be the largest perfect square dividing a. Then for a given d

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a \leq N \\ Q(a)=d}} 1 = \frac{6}{\pi^2 d^2}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a \leq N \\ Q(a)>d}} 1 = 1 - \frac{6}{\pi^2} \sum_{j=1}^{d-1} \frac{1}{j^2}.$$

Let f be squarefree and prime to d. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a \leq N \\ Q(a)=d \\ (a,f)=1}} 1 = \frac{6}{\pi^2 d^2} \prod_{p|f} \frac{p}{p+1}.$$

The proofs of these formulas are based on the following formula: $\lim_{N\to\infty} N^{-1} \sum_{a \leq N} \mu^2(a) = 6\pi^{-2}$ [cf. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Bd II, Teubner, Leipzig-Berlin, 1909, pp. 580 ff.].

P. Scherk (Saskatoon, Sask.).

Ward, Morgan. The maximal prime divisors of linear recurrences. Canad. J. Math. 6, 455–462 (1954).

Soit $W_0, W_1, \cdots, W_n, \cdots$ une suite récurrente à termes entiers rationnels et à relation de récurrence

$$x_{n+r} = P_1 x_{n+r-1} + P_2 x_{n+r-2} + \cdots + P_r x_n,$$

où les P_i sont, également, des entiers rationnels. Soit $f(z) = z^r - \sum_{i=1}^{r} P_i z^{r-i}$ le polynôme caractéristique de cette relation de récurrence, qu'on va supposer sans zéros multiples. Soient D le discriminant de $f(z)$, $\Delta(W)$ le discriminant de la matrice $\|W_{i+j-2}\|$ $(i, j = 1, 2, \cdots, r)$ d'ordre r,

$$\Delta(W) = DP_r \Delta(W), \quad f_q(z) = z^q - \sum_{i=1}^{q} P_i z^{q-i},$$

$$w(z) = \sum_{i=0}^{r-1} W_i f_{r-i-1}(z), \quad g(z) = z^r - \sum_{i=1}^{r} Q_i z^{r-i}$$

le résultant de $z - w(y)$ et de $f(y)$. Soient $L_0, L_1, \cdots, L_n, \cdots$ et $M_0, M_1, \cdots, M_n, \cdots$ les suites récurrentes satisfaisant respectivement aux relations $x_{n+r} = \sum_{i=1}^{r} P_i x_{n+r-i}$ et $x_{n+r} = \sum_{i=1}^{r} Q_i x_{n+r-i}$ avec les données initiales

$$L_0 = L_1 = \cdots = L_{r-2} = M_0 = M_1 = \cdots = M_{r-2} = 0,$$
$$L_{r-1} = M_{r-1} = 1.$$

Un entier rationnel positif est dit diviseur de la suite (W_n) s'il en divise quelque terme d'indice $\neq 0$. Un nombre premier p est dit un diviseur de (W_n) d'ordre $s < r$ s'il existe s termes consécutifs de (W_n) tous divisibles par p, sans qu'il existe $s+1$ termes de cette forme. Le premier p est dit un diviseur maximal de (W_n) s'il en est un d'ordre $r-1$. On constate que tout p ne divisant pas P_r est un diviseur maxi-

mal de (L_n), et il en est de même pour tout p ne divisant pas Q_r et pour la suite (M_n). On appelle rang de p dans (L_n) le plus petit indice $\rho>0$ tel que $L_\rho\equiv L_{\rho+1}\equiv\cdots\equiv L_{\rho+r-2}\equiv 0 \pmod p$, et on définit de même son rang dans (M_n).

L'auteur démontre les résultats suivants: I. La condition nécessaire pour qu'un nombre premier p ne divisant pas $\Delta(W)$ soit un diviseur maximal de (W_n) est que p ne divise pas Q_r et que son rang dans (M_n) divise son rang dans (L_n). II. Cette condition nécessaire est aussi suffisante si: a) $f(z)$ est irréductible et de degré impair; b) $p-1$ est premier au degré r de $f(z)$; c) $f(z)$ est irréductible (mod p). [Remarque de référent: la condition a) est inutile, car l'irréductibilité de $f(z)$ résulte de son irréductibilité (mod p), tandis que l'imparité de son degré (qui résulte, d'ailleurs, de la condition b) quand $p\neq 2$) n'intervient pas dans les raisonnements.]

M. Krasner (Paris).

Carlitz, L. The coefficients of the reciprocal of $J_0(x)$. Arch. Math. 6, 121–127 (1955).

Of the three theorems proved in this paper the following is representative. Let F_m, G_m denote polynomials in an arbitrary number of indeterminates with integral coefficients and let F_m, G_m each satisfy

$$(*)\qquad F_m\equiv F_{a_0}F_{a_1}^{p}F_{a_2}^{p^2}\cdots \pmod p,$$

where $m=a_0+a_1p+a_2p^2+\cdots$ $(0\leqq a_i\leqq p-1)$ and p is a prime $\geqq 2$. Then the polynomial $H_m=\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}^2F_rG_{m-r}$ also satisfies (*). Among the applications it is shown that the coefficients of the reciprocal of $J_0(x)$ satisfy a congruence of the same form as (*). Schur's congruence for the Legendre polynomial $P_m(x)$ is also derived. *A. L. Whiteman.*

Carlitz, L. Extension of a theorem of Glaisher and some related results. Bull. Calcutta Math. Soc. 46, 77–80 (1954).

In this note the author considers the numbers $A_r^{(k)}$ defined by

$$\prod_{r=1}^{p-1}(x-kp-r)=x^{p-1}-A_r^{(k)}x^{p-2}+\cdots+A_{p-1}^{(k)},$$

where k is an arbitrary integer and p is a prime >3. Generalizing some formulas of Glaisher [Quart. J. Pure Appl. Math. 31, 1–35 (1899), 321–353 (1900)] he derives a number of miscellaneous congruences for the $A_r^{(k)}$. A typical result states that $A_{2r}^{(k)}\equiv -pB_{2r}/2r \pmod{p^2}$ where $r\geqq 1$, $p>3$ and B_{2r} denotes the Bernoulli number in the even-suffix notation. He also derives the following extension of a formula of Ljunggren [Avh. Norske Vid. Akad. Oslo. I. 1947, no. 5 (1948); MR 9, 568]: If $q=p^r$ and $H_r=\binom{nq}{q}$, then

$$H_r-H_{r-1}\equiv -\tfrac{1}{2}n(n-1)q^3H_{r-1}B_{p-3} \pmod{n^2q^3p}.$$

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, Leonard. Sums of primitive roots of the first and second kind in a finite field. Math. Nachr. 12, 155–172 (1954).

A number $\beta\in GF(q)$, $\beta\neq 0$, is a primitive root of the field if $k=q-1$ is the smallest positive integer such that $\beta^k=1$. Let m be a fixed integer $\geqq 1$. Then for each $\gamma\in GF(q^m)$ there exists a linear polynomial $a(x)=\sum_{r=0}^{k}a_rx^{q^r}$ with minimum k such that $a(\gamma)=0$. The number γ is said to belong to $a(x)$. In particular if $a(x)=x^{q^m}-x$, then γ is called a primitive root of the second kind. In this paper the author discusses the number of solutions $\zeta_1,\cdots,\zeta_r\in GF(q^m)$ of the equation $\alpha=\alpha_1\zeta_1+\cdots+\alpha_r\zeta_r$, where $\alpha,\alpha_1,\cdots,\alpha_r$ are assigned and $\zeta_1,\cdots,\zeta_r$ are primitive roots of the second kind or simultaneously primitive roots of the first and second kinds. In the case in which the ζ's are restricted to be primitive roots of the second kind a simple exact formula for the number of solutions is found. The result resembles the singular series for the Goldbach problem. A number of related problems are also discussed in detail. The method used by the author is essentially that employed by him in two earlier papers [Duke Math. J. 19, 459–469 (1952); Trans. Amer. Math. Soc. 73, 373–382 (1952); MR 14, 357, 539].

A. L. Whiteman (Los Angeles, Calif.).

***Brauer, Alfred. Elementary estimates for the least primitive root.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 20–29. Academic Press Inc., New York, 1954. $9.00.

Using purely elementary methods the author obtains estimates for the least positive primitive root and the number of primitive roots (mod p^k) which are less than p. The following two results are typical. 1. Let p be a prime of the form $4n+1$ and k the number of different prime divisors of $p-1$. Put $r=2^k$. If g_0 is the smallest positive primitive root (mod p) then $g_0<p^{(r-1)/r}$. 2. For an odd prime p there are at least $\frac{1}{2}\varphi(p-1)$ primitive roots (mod p^k) which are less than p. *W. Simons* (Vancouver, B. C.).

Uchiyama, Saburô. Sur les polynômes irréductibles dans un corps fini. I. Proc. Japan Acad. 30, 523–527 (1954).

Let $q=p^n$, p prime >2, and consider polynomials $M(x)=x^m+c_1x^{m-1}+\cdots+c_m$, $c_i\in GF(q)$. Let $\pi_0(m;r,t)$ denote the number of polynomials $M(x)$ that are irreducible in $GF(q,x)$ and with assigned coefficients $c_1,\cdots,c_r$, $c_{m-t+1},\cdots,c_m$. The reviewer [Proc. Amer. Math. Soc. 3, 693–700 (1952); MR 14, 250] proved that

$$\pi_0(m;1,1)=\frac{1}{m}q^{m-2}+O(q^{m/2})\quad(m\to\infty).$$

The present paper contains proofs of the following two theorems:

$$\text{I)}\qquad \pi_0(m;0,2)=\frac{1}{m}q^{m-2}+O(q^{m/2})\quad(m\to\infty),$$

$$\text{II)}\qquad \pi_p(m;r,t)=\frac{1}{m}p^{m-r-t}+O(p^{\theta m})\quad(m\to\infty),$$

where $r+t\geqq 2$, $\max(r,t)<p$ and θ is a constant independent of p and m, $\frac{1}{2}\leqq\theta<1$. *L. Carlitz* (Durham, N. C.).

***Lamprecht, Erich. Gausssche Summen in endlichen Ringen und ihre Anwendungen.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 179–185. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

The author sketches the theory of Gaussian sums in finite commutative rings [for a detailed discussion see Lamprecht, Math. Nachr. 9, 149–196 (1953); MR 14, 942]. He also discusses briefly some applications of the sums.

L. Carlitz (Durham, N. C.).

Ballieu, Robert. Factorisation des polynomes cyclotomiques modulo un nombre premier. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 140–144 (1954).

The paper contains brief proofs of known theorems on the factorization of the cyclotomic polynomial to a prime modulus. *L. Carlitz* (Durham, N. C.).

Schwerdtfeger, H. Note on a theorem by J. A. Todd. J. London Math. Soc. **30**, 83–84 (1955).

The author proves the following theorem, a special case of which was found by J. A. Todd [same J. **20**, 204–209 (1945); MR **8**, 135]: Let $g(y)$ be a polynomial in $R[y]$, where R is an arbitrary number field, and define $f_m(x)=g(x^{2^m})$ $(m=1, 2, \cdots)$. If $f_m(x)$ is reducible in $R[x]$ whilst $f_{m-1}(x)$ is not, it follows that $m\leqq 2$. A stronger result holds if the powers of 2 are replaced by those of an odd prime. In that case it can be concluded that $m=1$. *W. Ledermann.*

*Hecke, Erich. **Vorlesungen über die Theorie der algebraischen Zahlen. 2te Aufl.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1954. viii +266 pp. DM 11.00.

Unaltered reproduction of the 1st edition [Leipzig, 1923].

Šafarevič, I. R. On the construction of fields with a given Galois group of order l^s. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 261–296 (1954). (Russian)

k étant un corps (de degré fini) de nombres algébriques, A. Scholz [Math. Z. **42**, 161–188 (1937)] et H. Reichardt [J. Reine Angew. Math. **177**, 1–5 (1937)] avaient donné une méthode de construction des extensions galoisiennes K/k ayant, comme groupe de Galois, un p-groupe donné. Mais leur méthode ne fournissait qu'une classe assez restreinte d'extensions et ne s'appliquait pas au cas $p=2$. L'auteur donne une autre méthode de construction d'extensions de cette forme, qui fournit une classe plus large d'extensions et s'applique au cas $p=2$. Une telle méthode semble indispensable pour résoudre le problème de construction des extensions à groupe de Galois résoluble donné.

Une surextension galoisienne K'/k d'une extension galoisienne K/k est dite centrale si le groupe de Galois $g=G_{K'/K}$ de K'/K est contenu dans le centre de celui $G=G_{K'/k}$ de K'/k. Alors, $\Gamma=G_{K/k}=G/g$ opère trivialement dans g et l'extension G de Γ par g définit une classe H de cohomologie (de dimension 2) de Γ dans g. Supposons que K contient le groupe E_p des racines p-ièmes de l'unité et que $K'=K(\mu^{1/p})$. Alors, K'/k est une surextension centrale de K/k si, et seulement si $\mu \in K$ est p-invariant dans K/k: autrement dit, on a, pour tout $\sigma \in \Gamma$, $\sigma\cdot\mu\equiv\mu \pmod{K^{(p)}}$, où $K^{(p)}$ est le groupe des puissances p-ièmes des éléments du groupe multiplicatif K^* de K. Si $z \in g$, $\lambda(z)=(z\cdot\mu^{1/p})\colon \mu^{1/p}$ est une racine p-ième de l'unité, et $z\to\lambda(z)$ est un isomorphisme de g sur E_p, qui applique H sur une classe de cohomologie $H'=H'(\mu)$ de Γ dans E_p. On a $H'(\mu)=H'(\nu)$ si et seulement si $\mu\equiv\nu \pmod{k^*K^{(p)}}$ et $\mu\to H'(\mu)$ est un homomorphisme du groupe multiplicatif J_p des $\mu \in K$ p-invariants dans le second groupe de cohomologie $\mathfrak{H}$ de Γ dans E_p, le noyau de cet homomorphisme étant $k^*K^{(p)}$. Soit $X \in C_p=J_p/k^*K^{(p)}$ et soit $H(X)=H(\mu)$, où $\mu \in X$. A. Scholz a montré que $X\to H(X)$ est un isomorphisme de C_p sur $\mathfrak{H}$ quand K/k satisfait, pour k suffisamment grand, aux conditions suivantes: 1) tous les facteurs premiers $\mathfrak{p}$ de p dans k et tous les idéaux à l'infini $\mathfrak{p}_\infty$ de k se décomposent complètement dans K/k; 2) tout diviseur premier $\mathfrak{q}$ dans k du discriminant $D_{K/k}$ de K/k se décompose dans K/k en facteurs premiers de degré 1 et il existe, pour $\mathfrak{q}$, un exposant $m(\mathfrak{q})\not\equiv 0 \pmod p$ tel que $N(\mathfrak{q})^{m(\mathfrak{q})}\equiv 1 \pmod{p^\lambda}$, où $N(\ )$ est la norme absolue. [Avertissement au lecteur: l'auteur emploie une terminologie inhabituelle, en appelant ordre ("poryadok") d'un idéal premier ce qu'on appelle habituellement son degré]. Une telle extension K/k est dite scholzienne d'exposant h.

K/k étant scholzienne d'exposant h, l'auteur détermine, d'abord, les conditions nécéssaires et suffisantes, auxquelles un $X \in C_p$ doit satisfaire pour qu'il existe un $\mu \in X$ tel que $K(\mu^{1/p})/k$ soit aussi scholzienne d'exposant h. Dans ce but, il définit certains invariants (χ, X) et $(X)_h$ de X, où χ parcourt les caractères de degré 1 de Γ (le manque de place empêche d'en donner ici la définition), et il montre que la condition nécessaire et suffisante de l'existence d'un tel $\mu \in X$ est l'égalité à 1 de tous ces invariants. En particulier, l'auteur montre par un contre-exemple l'inexactitude d'un résultat de T. Tannaka [Tôhoku Math. J. **43**, 252–260 (1937)], qui a cru démontrer qu'il existe toujours de tels $\mu \in X$, et indique où se trouve la faute de son raisonnement.

p^s étant l'ordre du groupe C, il résulte du critère précédent que la condition nécessaire et suffisante de l'existence de la surextension centrale scholzienne d'exposant h de K/k telle que g soit un p-groupe élémentaire d'ordre p^s est que les fonctions (χ, X) et (X) de X soient toutes identiquement égales à 1 sur C_p.

Voici une vague idée de la méthode de l'auteur pour construire les surextensions scholziennes K'/k de K/k de p-groupe donné G. Si U est un groupe, soit

$$U=U^{(0)}\supseteq U^{(1)}\supseteq\cdots\supseteq U^{(i)}\supseteq\cdots$$

la chaîne de ses sous-groupes telle que $U^{(i+1)}$ soit le groupe engendré par les puissances p-ièmes des éléments de $U^{(i)}$ et par les commutateurs de ces éléments avec ceux de U. Si U est nilpotent, on a, à partir d'un certain rang, $U^{(i)}=1$, et le plus petit indice c tel que $U^{(c)}=1$ sera dit, dans ce cas, la classe élémentaire de U. Il existe (et est unique à isomorphie près) le p-groupe $G_m^{(c)}$ à m générateurs et de classe élémentaire c, dont l'ordre est maximal, et tout p-groupe est un groupe quotient de quelque $G_m^{(c)}$, ce qui permet de réduire la construction considérée au cas $G=G_m^{(c)}$. La solution du cas $c=1$, quelque soit m, est donnée par la théorie des corps de classes. L'auteur suppose le problème résolu, quelque soit m, pour un $c\geqq 1$ donné. Alors, m étant donné d'avance, il montre que si n est suffisamment grand par rapport à m, on peut toujours construire, à partir d'une extension normale scholzienne (d'exposant h) $K_n^{(c)}/k$ de groupe $G_n^{(c)}$, une sous-extension normale scholzienne (de même exposant) K/k de $K_n^{(c)}/k$ de groupe $G_m^{(c)}$ et dont tous les invariants (χ, X) et $(X)_h$ soient identiquement égaux à 1. Comme un tel $K_n^{(c)}/k$ existe, pour tout n, en vertu de l'hypothèse de l'induction, on voit que la construction précédente fournit, pour tout m, une extension normale scholzienne (d'un exposant h donné d'avance) $K_m^{(c+1)}/k$ de groupe $G_m^{(c+1)}$, donc résout le problème pour $c+1$. Pour construire K/k à partir de $K_n^{(c)}/k$, l'auteur fixe n générateurs $\tau_1, \tau_2, \cdots, \tau_r$ de $G_n^{(c)}=G_{K_n^{(c)}/k}$, qui engendrent ce groupe, et considère les homomorphismes S de $G_n^{(c)}$ sur $G_m^{(c)}$, (où on a également fixé m générateurs $\sigma_1, \sigma_2, \cdots, \sigma_m$ engendrant $G_m^{(c)}$), tels que $S\cdot\tau_i=\sigma_i$ si $1\leqq i<m$ et $S\cdot\tau_i=\sigma_m$ ou $S\cdot\tau_i=1$ si $m\leqq i\leqq n$. Si N^S est le noyau de cet homomorphisme, soit K^S est le sous-corps de $K_n^{(c)}/k$, qui lui appartient et soit X une classe de p-invariance dans $K_n^{(c)}/k$. Si $H(X)$ est la classe de cohomologie de $G_n^{(c)}$ dans E_p, qui correspond à X, S applique $H(X)$ sur une classe de cohomologie de $G_m^{(c)}=G_n^{(c)}/N^S$ dans E_p, qui correspond à quelque classe de p-invariance X^S de K^S/k. Une classe de p-invariance X de $K_n^{(c)}/k$ et un caractère χ de degré 1 de $G_m^{(c)}$ étant fixés, (χ, X^S) est une fonction de S, et il en est de même pour $(X^S)_h$. L'auteur montre que, si n est assez grand, il existe des S de la forme précédente (dont il indique la construction) tels que, pour tous χ et X, (χ, X^S) et $(X^S)_h$ soient $=1$. *M. Krasner* (Paris).

Šafarevič, I. R. On extensions of fields of algebraic numbers solvable in radicals. Dokl. Akad. Nauk SSSR (N.S.) **95**, 225–227 (1954). (Russian)

Il est affirmé que, pour tout corps de degré fini k de nombres algébriques, il existe une extension K/k dont le groupe de Galois est un groupe résoluble G donné d'avance, et des indications sur la marche de la démonstration sont données, qui montrent qu'elle doit être une généralisation de celle du travail analysée ci-dessus. Comme dans ce travail, le problème est réduit au cas, où G est un groupe $G_{d,\tau}^{(c)}$ de type spécial tel que tout groupe résoluble, dont l'ordre n'aie comme diviseurs premiers que les nombres premiers donnés $p_1, p_2, \cdots, p_s$, soit un groupe facteur de quelque groupe de ce type. L'auteur définit la notion générale d'extension scholzienne et, dans le cas de telles extensions ayant un $G_{d,\tau}^{(c)}$ comme groupe de Galois, il définit certains invariants, dont il affirme que leur égalité à 1 est la condition nécéssaire et suffisante pour que l'extension considéré puisse être plongée dans sa surextension centrale de groupe $G_{d,\tau}^{(c+1)}$. D'autre part, comme dans le mémoire cité de l'auteur, pour tout δ il existe une constante $C(\delta)$ telle que, dans toute extension scholzienne de groupe $G_{d,\tau}^{(c)}$ avec $d \geqq C(\delta)$, il existe une sous-extension de groupe $G_{d,\tau}^{(c)}$, ayant tous ces invariants égaux à 1, d'où l'existence d'une extension K/k de groupe $G_{d,\tau}^{(c)}$ résulte par induction sur c.

Voici les définitions de l'auteur: h étant un entier divisible par $p_1 p_2 \cdots p_s$, K/k est dite scholzienne relativement à h si tous les diviseurs premiers du discriminant de K/k dans K sont de degré 1 dans K/k et ont leurs normes absolues $\equiv 1 \pmod h$ et si tous les diviseurs premiers de h dans k ainsi que tous les diviseurs premiers à l'infini de k se décomposent complètement dans K/k. Soit π l'ensemble $\{p_1, p_2, \cdots, p_s\}$; $G_{d,\tau}^{(c)}$ est le quotient $S_d/N_{d,\tau}^{(c)}$ du groupe libre S_d à d générateurs par son sous-groupe invariant $N_{d,\tau}^{(c)}$ défini par la récurrence suivante: $N_{d,\tau}^{(0)} = S_d$; $G_i/N_{d,\tau}^{(c-1)}$ étant un p_i-groupe de Sylow de $G_{d,\tau}^{(c-1)} = S_d/N_{d,\tau}^{(c-1)}$ et N_i étant le plus grand sous-groupe invariant de S_d contenu dans le groupe engendré par les puissances p-ièmes des éléments de G_i et par les commutateurs de ces éléments avec ceux de S_d, $N_{d,\tau}^{(c)}$ est l'intersection des N_i $(i=1, 2, \cdots, s)$. Quant à la définition des invariants, qui sont de trois types: $(\chi, X)^{(c)}$, $(X)_h^{(c)}$ et $(X)_\phi^{(c)}$, dont les deux premiers généralisent les invariants (χ, X) et $(X)_h$ du travail cité de l'auteur et dont le troisième est nouveau, elle est trop compliquée pour être donnée ici. *M. Krasner* (Paris).

Šafarevič, I. R. On the problem of imbedding fields. Dokl. Akad. Nauk SSSR (N.S.) **95**, 459–461 (1954). (Russian)

Dans la prémière partie de cette note, l'auteur énonce le résultat suivant: F et G étant deux groupes d'ordre fini, dont G est un p-groupe de classe $\leqq p$, soit $\mathfrak{G}$ un produit semi-direct de F par G, autrement dit une extension de F par G (au sens de Schreier), F pouvant opérer sur G d'une manière non-triviale, telle que sa classe de cohomologie soit l'unité du second groupe de cohomologie de F dans G. Alors, toute extension L d'un corps k de degré fini de nombres algébriques ayant F comme groupe de Galois peut être plongé dans une surextension K/k dont le groupe de Galois soit $\mathfrak{G}$ (il s'agit, bien entendu, d'immersion identifiant l'homomorphisme naturel de $\mathfrak{G}$ sur F avec l'homomorphisme naturel du groupe de Galois de K/k sur celui de L/k). De quelques indications sur la marche de démonstration données par l'auteur, il résulte la dépendance de cette démonstration par rapport à la théorie ésquissée dans sa note analysée ci-dessus et le rôle important joué par les invariants définis dans cette note. Dans la seconde partie de la présente note, l'auteur formule, à l'aide de ces invariants, la condition nécessaire et suffisante pour qu'une extension scholzienne $k(\alpha_1^{1/p}, \alpha_2^{1/p}, \alpha_3^{1/p})/k$ soit immersible dans une surextension dont le groupe de Galois soit le quotient du produit libre de 3 groupes cycliques d'ordre p (ou p est un nombre premier) par le troisième terme de sa suite centrale descendante. Il en résulte qu'une condition nécessaire d'immersibilité, donnée par B. N. Delone et D. K. Faddeev [Mat. Sb. N.S. **15(57)**, 243–284 (1944); MR **6**, 200] et H. Hasse [Math. Nachr. **1**, 40–61 (1948); MR **10**, 426], n'est pas suffisante, contrairement à une hypothèse de Hasse, fait qui a été montré récemment aussi par D. K. Faddeev [Dokl. Akad. Nauk SSSR (N.S.) **94**, 1013–1016 (1954); MR **15**, 938] par un exemple plus particulier. *M. Krasner* (Paris).

Šafarevič, I. R. On an existence theorem in the theory of algebraic numbers. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 327–334 (1954). (Russian)

Soient k un corps (de degré fini) de nombres algébriques, K une extension de degré fini de k contenant une racine p-ième primitive ξ de l'unité (où p est premier), G le groupe de Galois de K/k, $(\alpha/\mathfrak{q})$ (où $\alpha \in K$ et où $\mathfrak{q}$ est un idéal de K premier à α) le symbole de Legendre d'exposant p dans K. Soit, pour tout $\sigma \in G$, $u(\sigma)$ le reste (mod p) tel que $\sigma \cdot \xi = \xi^{u(\sigma)}$.

L'auteur démontre le théorème suivant: si $p \neq 2$, soient $\zeta(\sigma)$ une fonction définie sur G et à valeurs dans le groupe des racines p-ièmes de l'unité telle que $\zeta(\sigma^{-1}) = \zeta(\sigma)^{u(\sigma)}$, $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_r$ des idéaux de K copremiers deux à deux et premiers à p, $\xi_1, \xi_2, \cdots, \xi_r$ des racines p-ièmes donnés de l'unité. Alors, il existe une infinité des $\alpha \in K$ tels que: 1) α est premier à tous les $\mathfrak{q}_i$ $(i=1, 2, \cdots, r)$; 2) si un idéal premier $\mathfrak{p}$ de K divise α, aucun conjugué de $\mathfrak{p}$ par rapport à k ne le divise; 3) pour tout $i=1, 2, \cdots, r$, on a $(\alpha/\mathfrak{q}_i) = \xi_i$; 4) pour tout idéal premier $\mathfrak{p}$ de K tel que $\mathfrak{p}/\alpha$, on a $(\alpha/\sigma \cdot \mathfrak{p}) = \zeta(\sigma)$. Si $p=2$, le même résultat est valable avec les hypothèses supplémentaires suivantes: a) les $\mathfrak{q}_i$ $(i=1, 2, \cdots, r)$ sont premiers au discriminant de K/k; b) tous les diviseurs premiers de 2 dans k et tous les idéaux premiers à l'infini de k se décomposent complètement dans K/k; c) D_σ désignant le discriminant par rapport à k du sous-corps K_σ de K appartenant au cycle de $\sigma \in G$, on a, pour tout ensemble $\{\sigma_1, \sigma_2, \cdots, \sigma_s\}$ d'automorphismes involutifs $(\sigma_j^2 = 1)$ de K/k tel que $D_{\sigma_1} D_{\sigma_2} \cdots D_{\sigma_s}$ soit un carré parfait dans K, $\zeta(\sigma_1)\zeta(\sigma_2)\cdots\zeta(\sigma_s) = 1$.

Idée de démonstration pour $p \neq 2$. On construit une suite dénombrable $\pi_1, \pi_2, \cdots, \pi_n, \cdots$ d'entiers p-hyperprimaires de K telle que les (π_n) soient des idéaux premiers de K non-conjugués deux à deux par rapport à k, satisfaisant aux conditions: 1) pour tout $i=1, 2, \cdots, r$, et pour tout n, $(\pi_n/\mathfrak{q}_i) = \xi_i^{1/2}$; 2) $\pi_1, \pi_2, \cdots, \pi_{n-1}$ étant choisis, π_n est assujetti, en plus, aux conditions $(\pi_n/\sigma \cdot \pi_m) = (\pi_m/\sigma \cdot \pi_m)^{-1}\zeta(\sigma)$ (σ parcourt les éléments de G autres que l'unité, $m = 1, 2, \cdots, n-1$). L'existence d'une telle suite est assurée par la loi généralisée de progression arithmétique. Comme $(\pi_n/\sigma \cdot \pi_n)$ ne prend qu'un nombre fini de valeurs, il existe une infinité de couples (m, n), $m < n$, tels que, pour tout $\sigma \in G$, on ait $(\pi_m/\sigma \cdot \pi_m) = (\pi_n/\sigma \cdot \pi_n)$. Si l'on pose $\alpha = \pi_m \pi_n$, les égalités $(\alpha/\mathfrak{q}_i) = \xi_i$ $(i=1, 2, \cdots, r)$ et $(\alpha/\sigma \cdot \pi_n) = \zeta(\sigma)$ sont automatiquement satisfaits, tandis que l'égalité $(\alpha/\sigma \cdot \pi_n) = \zeta(\sigma)$ s'établit facilement à l'aide de la loi de réciprocité.

La démonstration pour $p=2$ est beaucoup plus compliquée. *M. Krasner* (Paris).

Fröhlich, A. On the absolute class-group of Abelian fields. J. London Math. Soc. **29**, 211–217 (1954).
Fröhlich, A. A remark on the class number of Abelian fields. J. London Math. Soc. **29**, 498 (1954).

Let K be a finite Abelian extension of degree l (l a prime) over the rational field P. Let $\mathfrak{M}=\{M\}$ be the set of all normal, relatively non-ramified (except $\mathfrak{p}_\infty$) Abelian extensions of K whose degree is a power of l. In case there is no proper absolutely Abelian extension M in $\mathfrak{M}$, the author determines explicitly the Galois group $\Gamma(\bar{K})$ of $\bar{K}$ over $\mathfrak{M}$, where $\bar{K}$ is the maximal field in $\mathfrak{M}$ such that the Galois group of $\bar{K}/K$ lies in the center of $\Gamma(\bar{K})$, by means of the conductor $f(K)=\prod_{i=1}^{m}\mathfrak{p}_i^{\rho_i}\cdot\mathfrak{p}_\infty^{\rho}$ over P and the set of cross coefficients $\{[\mathfrak{p}_i,\mathfrak{p}_j]\}$, which was defined in another paper [Proc. London Math. Soc. (3) **4**, 235–256 (1954); MR **16**, 116]. Using this result the author proves that K has an absolute class number (in the narrow sense) relatively prime to l if and only if one of the following conditions is satisfied: (i) the conductor $f(K)$ is a prime power; (ii) $f(K)=\mathfrak{p}_1^{\rho_1}\cdot\mathfrak{p}_2^{\rho_2}$ and K is the product of two cyclic fields K_1, K_2 with $f(K_i)=\mathfrak{p}_i^{\rho_i}$ (and with other additional properties on the cross coefficients); (iii) $K=M_\lambda\cdot P(\sqrt{-1})\cdot K_1$ where $f(K_1)=\mathfrak{p}_1\neq 2, 2$ is not a quadratic residue mod $\mathfrak{p}_1$, and M_λ is a real cyclic field with $f(M_\lambda)=2^\lambda\cdot\mathfrak{p}_\infty^{\rho}$; (iv) $f(K)=\mathfrak{p}_1^{\rho_1}\cdot\mathfrak{p}_2^{\rho_2}\cdot\mathfrak{p}_3^{\rho_3}$ and K is the product of three cyclic fields with $f(K_i)=\mathfrak{p}_i^{\rho_i}$ ($i=1, 2, 3$) (with other properties on the cross coefficients). Some immediate consequences of this theorem are given. The proofs depend largely on the results of the above-mentioned paper.

Y. Kawada (Tokyo).

Fröhlich, A. On the absolute class-group of Abelian fields. II. J. London Math. Soc. **30**, 72–80 (1955).

The author deals with problems analogous to those in his earlier paper [see the preceding review]; but here he considers the class number h_w in the wider sense for a finite Abelian extension K of the prime field P whose degree is a power of a prime l. Since $h_w=h$ (the class number of K in the narrow sense) if $l\neq 2$ or $l=2$ and K is complex, here a real field K of degree 2^n is considered, and the author gives necessary and sufficient conditions that h_w be odd. They are: (i) every ideal norm of K is congruent modulo the conductor $f(K)$ to a principal ideal norm of K; (ii) an equality containing the number n of distinct rational prime divisors of $f(K)$ and the cross coefficients. In particular, h_w is even if $n>4$ and also if $n=3$ and -1 is a total norm residue of K. Some corollaries are given. *Y. Kawada* (Tokyo).

Fröhlich, A. The generalization of a theorem of L. Rédei's. Quart. J. Math., Oxford Ser. (2) **5**, 130–140 (1954).

The author considers the structure of l-class group in the narrow sense of an absolutely cyclic field K of degree l^ν (l a prime). Let $\mathfrak{p}_i$ ($i=1, \cdots, m$) be the distinct prime divisors of the conductor $f(K)$, of ramification orders l^{ν_i} ($\nu_i\geqq\nu_{i+1}$). Let A be the group of ideals in K, I be the group of principal ideals, generated by totally positive numbers in K, and σ be a fixed generator of the cyclic Galois group of K/P. The integers $[\mathfrak{p}_i,\mathfrak{p}_j]$ are defined by $\chi_i(\mathfrak{p}_j)=\epsilon(l^{\nu-\nu_i}[\mathfrak{p}_i,\mathfrak{p}_j])$ (for $i\neq j$) ($0<[\mathfrak{p}_i,\mathfrak{p}_j]\leqq l^{\nu_i}$) and by $[\mathfrak{p}_i,\mathfrak{p}_i]=l^{\nu_i}$. Here ϵ is a fixed primitive l^ν-th root of unity, $\epsilon(a)=\epsilon^a$, K belongs to a cyclic group of characters generated by $\chi=\prod_{i=1}^{m}\chi_i^{y_i}$, $(y_i, l)=1$, $y_1=1$, where χ_i is a character of order l^{ν_i} with a power of $\mathfrak{p}_i$ as conductor ($i=1, \cdots, m$). It is known that $A/A^{(1-\sigma)}I$ has the invariants $(l^{\nu_1}, \cdots, l^{\nu_m})$. The main results of this paper is the following: let $T=\begin{pmatrix}U\\L\end{pmatrix}$, $U=(u_{ij})$, $L=(l_{ij})$ ($i, j=1, \cdots, m$), $u_{ij}=-[\mathfrak{p}_j,\mathfrak{p}_i]s_i$ (for $i\neq j$), $u_{ii}=\sum_{j=1}^{m}s_j[\mathfrak{p}_j,\mathfrak{p}_i]$, $l_{ij}=0$ ($i\neq j$), $l_{11}=1$, $l_{ii}=l^{\nu_i}$ ($i>1$). Then the invariants of $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$ are elementary divisors other than 1 in the ring of rational l-integers of the matrix T. In case of a quadratic field K, $A^{(1-\sigma)^\nu}I=A^{\sigma}I$ and the above theorem implies a result of Rédei [J. Reine Angew. Math. **171**, 55–60 (1934)]. In case $\nu=1$ let d_l be the number of invariants of A/I which are multiples of l ($l\neq 2$). Then $2(m-1)-r\leqq d_l\leqq(l-1)(m-1)-(l-2)r$, where r is the rank of U. This is a generalization of a theorem of Inaba [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **4**, 61–115 (1940); MR **2**, 147]. There are other results which are related to results of Rédei and Reichardt [J. Reine Angew. Math. **170**, 69–74 (1933)]. *Y. Kawada* (Tokyo).

Fröhlich, A. A note on the class field tower. Quart. J. Math., Oxford Ser. (2) **5**, 141–144 (1954).

Let K be an absolute cyclic field of prime degree l. Let K_{n+1} be the maximal non-ramified (except $\mathfrak{p}_\infty$) class field of K_n whose relative degree is a power of l ($K_0=K$). Let $m(K)$ be the number of distinct rational prime divisors of the discriminant of K. The author proposes to prove that (i) $K_{n+1}\neq K_n$ if $m>g(n)$, (ii) for any given l and $m\leqq g(n)$ there exists K of degree l with $m(K)=m$ such that $K_{n+1}=K_n$ (for suitable $g(n)$). In this paper the author proves the case $n=1$ for $g(1)=3$ using the results of his earlier papers [see the three preceding reviews]. *Y. Kawada* (Tokyo).

Gut, Max. Relativquadratische Zahlkörper, deren Klassenzahl durch eine vorgegebene ungerade Primzahl teilbar ist. Comment. Math. Helv. **28**, 270–277 (1954).

Let k be a field of algebraic numbers which contains a primitive pth root of unity ζ (where p is an odd prime) and in which p has a prime factor $\mathfrak{p}$ of absolute degree 1. The author constructs infinitely many quadratic extensions K/k of k whose class numbers are divisible by p; moreover, for each K, the author constructs explicitly a cyclic unramified extension of degree p of K. The construction is as follows. Let γ be an integer of k which is relatively prime to $2p$ and which is a quadratic rest modulo $\mathfrak{p}$; set $\lambda=1-\zeta$. Then the fields K are of the form $k(u)$, with $u^2-(\lambda^{2p}\gamma+2)u+1=0$. For each K, the extension $K(u^{1/p})/K$ is cyclic of degree p and unramified. The proof depends on the study of the properties of the polynomials G_p defined by the condition that $G_p(z+z^{-1})=z^p+z^{-p}$: if we set $z=u^{1/p}$, $x=z+z^{-1}$, then $G_p(x)=\lambda^{2p}\gamma+2$. The crucial part of the proof consists in establishing that a root of this equation cannot be of degree either 1 or 2 over k. *C. Chevalley* (New York, N. Y.).

Watson, G. L. Representation of large numbers by cubic forms in seven positive integral variables. Proc. London Math. Soc. (3) **2**, 311–325 (1952).

Let

$$f(u, u, \cdots, t)=a(u_1^3+u_2^3)+b(v_1^3+v_2^3)+c(w_1^3+w_2^3)+dt^3.$$

The author proves the following results. 1) If $a=b=c=1$ and for any positive integral d, the form f represents all sufficiently large positive integers for positive integer values values of the variables. 2) If $a=b=d=1$, the form has the same property provided the positive integer c is not divisible by 3, 7 or 19. 3) Sufficient conditions for f to represent all

sufficiently large integers for positive integer values of the variables are: (i) $(3, d)=1$, $(42d, abc)=1$; (ii) if any prime p divides all three of a, b, c, then p^2 divides none of them; and (iii) no prime $\equiv 1 \pmod 6$ divides more than one of a, b, c. *B. W. Jones* (Boulder, Colo.).

Godwin, H. J. On the inhomogeneous minima of certain convergent sequences of binary quadratic forms. Quart. J. Math., Oxford Ser. (2) **5**, 28–46 (1954).

Let $f(x, y)$ be an indefinite binary quadratic form and $M(a, b)$ denote $\min f(x, y)$ for all x, y congruent mod 1 to the given numbers a and b respectively. Let $M_1(f)$ be the least upper bound of $M(a, b)$ for all values of a and b. In this paper M_1 is evaluated for the forms

$$\left(x+\frac{q_{n+1}}{q_n}y\right)\left(x-\frac{q_{n-1}}{q_n}y\right),$$

where q_n is the denominator of the nth convergent in the continued fraction expansion of $\frac{1}{2}(1+\sqrt{5})$. Also, M_1 is evaluated for the forms $x^2-p_n^2q_n^{-2}y^2$, where p_n/q_n is the nth convergent in the continued fraction expansion of $\sqrt{2}$. The limiting forms are, in the respective cases, x^2+xy-y^2 and x^2-2y^2 and the limiting values of M_1 are easily determined. *B. W. Jones* (Boulder, Colo.).

Ehrhart, Eugène. Une généralisation du théorème de Minkowski. C. R. Acad. Sci. Paris **240**, 483–485 (1955).

A short proof of the following analogue of Minkowski's fundamental theorem in 2 dimensions is given: Let K be a plane closed convex body with area $\geqq 4.5$ and let Λ denote any lattice of determinant 1 having a point at the centroid G of K. Then there exist at least two points of Λ, other than G, in K.

The result is best possible in two ways. (i) The constant 4.5 is exact, and is attained when K is a triangle. (ii) When K is a triangle, there is a lattice Λ of determinant 1 having only two points, other than G, in K. In a footnote, the author states a somewhat similar result for convex solids of revolution in 3 dimensions. *J. H. H. Chalk* (London).

Poitou, Georges. Sur l'approximation des nombres complexes par les nombres des corps imaginaires quadratiques dénués d'idéaux non principaux, particulièrement lorsque vaut l'algorithme d'Euclide. Ann. Sci. Ecole Norm. Sup. (3) **70**, 199–265 (1953).

Let m be a positive integer, and let x be a complex number not in the quadratic field $K=R(\sqrt{-m})$ of discriminant $-D$. Let $C(x)$ designate the limit superior of $|q(qx-p)|^{-1}$ for all integers p, q of K with $q\neq 0$. The set of numbers $C(x)$ is called the Markov spectrum of K, and the inf C_D of the spectrum is called the Hurwitz constant of K. The present paper is an elaborate investigation of the spectra and Hurwitz constants for certain small values of D. The results obtained are as follows: (a) For $D=3$, the numbers $13^{1/4}$, 2, $(32\cdot 3^{1/2}/13)^{1/2}$ are isolated points of the spectrum; every other value is greater than 2.070068. The number

$$((28+16\cdot 3^{1/2})/13)^{1/2}=2.0701\cdots$$

is a point of accumulation in the spectrum. (b) $C_{11}=\frac{1}{2}\sqrt{5}$. [This had been announced earlier; cf. Descombes and Poitou, C. R. Acad. Sci. Paris **231**, 264–266 (1950); MR **12**, 162.] It is an isolated point, the next value being larger than 1.21. (c) $C_{19}=1$. (d) $C_{20}\leqq 1$. (e) $C_{43}\leqq\sqrt{(5/11)}$.

The basic tools in the investigation are the notions of a sequence of best approximations and of a best regular sequence. If x non-ε K, a fraction p_0/q_0 in K is called a convergent (réduite) to x if there is no simpler fraction closer to x, i.e., if $\min(|qx-p|)=|q_0x-p_0|$, the minimum being taken over all integers $p, q \varepsilon K$ such that $|q|\leqq|q_0|$. Choosing one representative from each set of equivalent convergents, and arranging these representatives by the size of $|q|$, we obtain the sequence of best approximations to x. Consequtive convergents p/q and p'/q' to x are said to be adjacent if $d=pq'-p'q$ is a unit of K; while such convergents are always adjacent in the rational case, they need not be in other fields. This was implicit in the work of J. W. S. Cassels, W. Ledermann and K. Mahler [Philos. Trans. Royal Soc. London. Ser. A. **243**, 585–626 (1951); MR **13**, 323] in the case $D=4$. As a preliminary step in the present work, the author finds all the permissible values of d for $D=3, 4, 7, 8, 11, 19$, in which cases the integers of the fields form Euclidean rings. The existence of a Euclidean algorithm in these cases also makes it possible to assert that, given any irreducible fraction p_0/q_0 in K, there is a p/q adjacent to p_0/q_0 such that $|qx-p|<|q_0x-p_0|$. If this fact is applied repeatedly, and at each stage the simplest fraction is chosen, a best regular sequence of approximations results, and this in turn yields a regular continued fraction expansion for x, the partial quotients being integers of K. In the case $D=3$, the only possible values of d are the units of the field; it is in part this apparently fortuitous circumstance that makes possible the relatively precise results listed above.

The author was apparently unaware of the work of Cassels, Ledermann and Mahler cited above, as well as that of the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A. **55**, 415–426, 526–535, 536–545 (1952); MR **14**, 850] where the continued-fraction expansion just mentioned was considered in the Gaussian field. In addition to these papers and the copious references provided by Poitou, the interested reader should examine the recent paper by R. Mönkemeyer [Math. Nachr. **11**, 321–344 (1954); MR **16**, 223], in which similar ideas occur. *W. J. LeVeque.*

Cassels, J. W. S. Simultaneous Diophantine approximation. J. London Math. Soc. **30**, 119–121 (1955).

Let C denote the upper bound of those positive constants c for which the inequalities

$$\left|\alpha-\frac{p}{r}\right|\leqq cr^{-3/2};\quad \left|\beta-\frac{q}{r}\right|\leqq cr^{-3/2}$$

for all couples of real numbers α, β have an infinity of integer solutions $r>0$, p, q. It is shown that $C\leqq 7/2$. First it is remarked that the critical determinant of the domain $|X|\max(Y^2, Z^2)\leqq 1$ is C and then that for any three linear forms L_i $(i=1, 2, 3)$ giving a critical lattice for $|XYZ|\leqq 1$, so that they have determinant 7, the related forms

$$M_1=L_1,\quad M_2=\tfrac{1}{2}(L_2+L_3),\quad M_3=\tfrac{1}{2}(L_2-L_3)$$

have determinant $7/2$ and give an admissible lattice for the above mentioned domain. The proof of the first remark depends on a modification of an argument due to Davenport and Mahler [Duke Math. J. **13**, 105–111 (1946); MR **7**, 506]. *J. F. Koksma* (Amsterdam).

ANALYSIS

*Ważewski, T., in Zusammenarbeit mit K. Borsuk, K. Kuratowski, F. Leja, E. Marczewski, S. Mazur, J. G. Mikusiński, W. Orlicz, R. Sikorski, und W. Ślebodziński. **Der Einfluss moderner mathematischer Methoden auf die klassischen Theorien der Mathematik.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 45–68. Deutscher Verlag der Wissenschaften, Berlin, 1954.

Survey paper divided into the following sections: topology; theory of measure and integration; functional analysis. Each section is further subdivided into paragraphs treating separate topics and giving references.

Shenton, L. R. A determinantal expansion for a class of definite integral. II. Proc. Edinburgh Math. Soc. (2) 10, 78–91 (1954).

In a former paper [same Proc. (2) 9, 44–52 (1953); MR 15, 781] the author obtained a determinantal expansion of

$$\int_a^b \frac{A(x)B(x)}{C(x)}w(x)\,dx,$$

where (a, b) is a finite interval, and $w(x)$ is a non-zero weight function. He now states more general conditions under which his expansion holds, and considers also semi-infinite and infinite intervals with weight functions of the forms

$$A(x)\exp(-x),\quad A(x)\exp(-x^2).$$

He applies his results to the following integrals,

$$\int_{-1}^{1}(x^2+2px+q)^{-1}(1-x^2)^{-1/2}\,dx,\quad \int_0^\infty \frac{e^{-ax}x^{b-1}}{1-e^{-x}}\,dx,$$

$$\int_0^\infty e^{-xt}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right)dx,\quad \int_{-\infty}^{\infty}(x^2+2ax+b)^{-1}\exp(-\tfrac{1}{2}x^2)\,dx,$$

and points out the relation of his expansions to the expansions in continued fractions. *A. Erdélyi.*

Roth, K. F. On irregularities of distribution. Mathematika 1, 73–79 (1954).

If $P_1, \cdots, P_N$ are N points in $(0\leqq x\leqq 1, 0\leqq y\leqq 1)$ and $S(x', y')$ is the number of them in $(x<x', y<y')$, then $\int_0^1\int_0^1[S(x,y)-Nxy]^2dxdy>c\log N$. It follows that there exists (x', y'), $|S(x', y')-Nx'y'|>c'\sqrt{\log N}$, and also that

$$F_n(A)>c''\sqrt{\log n},$$

greatly improving the result $F_n>k\log\log n/\log\log\log n$ of van Aardenne-Ehrenfest [Nederl. Akad. Wetensch., Proc. 52, 734–739 (1949); MR 11, 336] on the impossibility of a "just" distribution. *H. D. Ursell* (Leeds).

Hlawka, Edmund. Über einen Satz von van der Corput. Arch. Math. 6, 115–120 (1955).

Let G be a compact group, with Haar measure so normalized that the measure of G is 1. Let $D_j(x)$, for $x\in G$ and $j=0, 1, \cdots$, be the irreducible unitary representations of G, with notation chosen so that D_0 is the identity representation. A sequence $f=\{x_i\}$ of elements of G is said to be equidistributed on G if for each continuous function ϕ on G, $\lim_{N\to\infty}M_N(\phi, f)=M(\phi)$, where $M_N(\phi, f)=N^{-1}\sum_{i=1}^{n}\phi(x_i)$, $M(\phi)=\int_G\phi\,dx$. The known extension [see B. Eckmann, Comment. Math. Helv. 16, 249–263 (1943); MR 6, 146] of Weyl's criterion asserts that f is equidistributed on G if and only if $\lim_{N\to\infty}M_N(D_i, f)=0$ for each $j>0$. The author calls a sequence f weakly uniformly equidistributed if for each continuous function ϕ,

$$\lim_{N\to\infty}\limsup_{H\to\infty}\frac{1}{H}\sum_{r=0}^{H}|M_N(\phi, f_{rN})-M(\phi)|=0,$$

where $f_k=\{x_{i+k}\}$. It is easily seen that such a sequence is equidistributed on G.

J. G. van der Corput has shown [Acta Math. 56, 373–456 (1931)] that if $\{x_i\}$ is a real sequence such that $\lim_{i\to\infty}(x_{i+1}-x_i)=\alpha$, where α is irrational, then $\{x_i\}$ is equidistributed (mod 1). Generalizing this, the present author proves the following theorem: if $f=\{y_i\}$ is weakly uniformly equidistributed on G, and if $f'=\{x_i\}$ has the property that $\lim_{i\to\infty}y_{i+1}^{-1}x_{i+1}x_i^{-1}y_i=e$ (the unit element of G), then f' is also weakly uniformly equidistributed on G. The theorem has content, since it is shown that almost all sequences are weakly uniformly equidistributed on G. *W. J. LeVeque* (Ann Arbor, Mich.).

Mambriani, Antonio. Su i prodotti delle derivazioni definite, d'ordine qualsiasi. Riv. Mat. Univ. Parma 5, 209–215 (1954).

Let $D^\mu f(x)$ denote the Riemann-Liouville derivative of f, of order μ, with origin x_0 (the same for all derivatives mentioned). The author's main result is that if n is a positive integer and $\Re(\nu)<n$, then

$$D^\mu\{D^\nu f(x)\}=D^{\mu+\nu}f(x)-\sum_{k=1}^{n}[D^{\nu-k}f(x)]_{x=x_0}\frac{(x-x_0)^{-\mu-k}}{(-\mu-k)!},$$

so that $D^\mu\{D^\nu f(x)\}=D^{\mu+\nu}f(x)$ provided either $\Re(\nu)<0$ or μ is a nonnegative integer. In particular $(\mu=-\nu, \Re(\nu)\geqq 0, \nu\neq 0)$, the following generalized Taylor formula holds:

$$f(x)=\sum_{k=1}^{n}[D^{\nu-k}f(x)]_{x=x_0}\frac{(x-x_0)^{\nu-k}}{(\nu-k)!}+D^{-\nu}\{D^\nu f(x)\},$$

$\Re(\nu)<n$. [Cf. the Copenhagen thesis of P. W. Marke, 1942; MR 8, 66.] *R. P. Boas, Jr.* (Evanston, Ill.).

Mirak'yan, G. M. On a convergent process of approximation of continuous functions. Akad. Nauk Armyan. SSR. Dokl. 16, 33–37 (1953). (Russian. Armenian summary)

Let $m_k\uparrow\infty$ be integers, $n_k\uparrow\infty$ positive numbers, such that $(m_k/n_k)^{1/2}\to r>0$. If $0<\rho<r$ and $f(x)$ is continuous on $[0, \rho]$, then

$$\sum_{k=0}^{m}(n^k/k!)x^{2k}e^{-n^2x^2}f\{(k/n)^{1/2}\}\to f(x)$$

uniformly on $[0, \rho]$. *R. P. Boas, Jr.* (Evanston, Ill.).

Calculus

*Menger, Karl. **Calculus, a modern approach. 2d. ed.** The Bookstore, Illinois Institute of Technology, Chicago, 1953. xxiv+304 pp. (mimeographed). $4.85.

This is a calculus textbook for undergraduates incorporating innovations in notation and terminology which the author considers necessary and sufficient for the avoidance

of ambiguities and errors associated with more conventional notations. The following is a quotation from the preface: "In this book we define a variable quantity in terms of the concepts of class and number. We develop a strictly deductive theory of variable quantities. We describe in detail the application of the results to physical science, where variable quantities are of paramount importance. We insist on a clearcut distinction between a variable quantity and a Weierstrass variable. We attribute some of the difficulties which Russell left unsolved, and some of the confusion in the mathematical literature, to a lack of such a distinction. We deny that the theory of variable quantities as used in science was arithmetized by Weierstrass, and claim that it cannot be arithmetized. We outline a postulational treatment of the theory of pure functions. Finally, we bring out an arithmetico-analytical parallelism between the mensuration of objects and the functional relation of variables—unit and independent variables being analogues."

By a "variable quantity" the author means a real-valued function defined over an arbitrary set, and by a "function" he means a real-valued function defined over a set of real numbers. Since this is a text-book for undergraduates, he does not give a completely rigorous presentation of his system. Aside from difficulties related to division by variable quantities which occasionally assume the value zero, he makes out a good case for the sufficiency of his system.

The reviewer could find no refutation of the conventional use of what the author calls the Weierstrass variable and what the other textbook writers call "x", according to which the later letters of the alphabet such as x, y, z are used for variables, and the earlier letters such as a, b, c for constants. In this notation $f(x)$ represents a function and $f(a)$ a functional value. If the author's letter I for the identity function is replaced throughout by x, his notation differs from the usual one principally by using $\int_a^b f$ instead of $\int_a^b f(x)dx$, a minor improvement. He does not discuss the possibility of avoiding confusion between the notions of "variable" and "variable quantity" by the conventional device of calling a variable quantity a real-valued function. He makes out a good case for the sufficiency of his system, and a less good case for its necessity. *O. Frink.*

Theory of Sets, Theory of Functions of Real Variables

***Szőkefalvi-Nagy, Béla. Valós függvények és függvénysorok. [Real functions and series of functions.]** Tankönyvkiadó, Budapest, 1954. 307 pp. 42.50 ft.

This is a text book for university students. The level of mathematical sophistication expected of the reader is only slightly below that for the well-known book by F. Riesz and the author [Leçons d'analyse fonctionnelle, Akad. Kiadó, Budapest, 1952; MR 14, 286]. The two books are also comparable in content and with respect to their high quality of exposition. Roughly speaking it might even be said that the present book is an expansion of the first part of Riesz-Nagy. The table of contents is as follows: I. Sets. II. Continuous functions. III. Differentiability. IV. Interval functions. The Riemann integral. V. The Lebesgue integral. VI. The Stieltjes integral and its generalizations. VII. Quadratically integrable functions. VIII. The convergence of Fourier series. IX. The summation of Fourier series.

P. R. Halmos (Chicago, Ill.).

Rado, R. Partial well-ordering of sets of vectors. Mathematika 1, 89–95 (1954).

G. Higman [Proc. London Math. Soc. (3) 2, 326–336 (1952); MR 14, 238] considered (quasi-) ordered sets S satisfying this finite basis property: Each nonvoid $A \subseteq S$ contains a finite set B so that for each $a \in A$ there is some $b \in B$ satisfying $b \leqq a$. Such a set S is called by the author "partially well-ordered" [p.w.o.]. Let α be any ordinal number and $W(\alpha)$ the set of ordinals $\leqq \alpha$; for any set S let $W_*(\alpha)$ be the set of all uniform mappings of $W(\alpha)$ into S; let $W_*(<\alpha) = \bigcup W_*(\xi)$ $(\xi < \alpha)$. The elements of $W_*(\alpha)$ are called "vectors over S of length α". Let $x, y \in W_*(<\alpha)$; then according to Higman one defines for any ordered S the quasi-order so that $x \leqq y$ if and only if the sequence x is embeddable into the sequence y so that no term of x exceeds the corresponding term of y. Let $V_*(\alpha)$ be the set of all $e \in W_*(\alpha)$ such that the set of values of e is finite. In order that $W_*(\omega)$ be p.w.o., it is necessary and sufficient that the following condition (α) holds: If $a_{r_s} < a_{r,s+1}$ $(r < s < \omega)$, then there are numbers r, s, t such that $r < s < t < \omega$, $a_{rs} < a_{st}$ (Th. 3.). A counter-example of the condition (α) is given (Th. 2). If $V_*(\alpha)$ is p.w.s., then also $V_*(<\alpha\omega)$ (Th. 4). $V_*(<\omega^3)$ is p.w.o. (Th. 5). *G. Kurepa* (Zagreb).

Zink, Robert E. Direct unions of measure spaces. Duke Math. J. 22, 57–74 (1955).

Placing appropriate finiteness and σ-finiteness conditions on the spaces that enter, the author defines the concept of the direct union of a family $\{Y_x\}$ of measure spaces, where the index x ranges over a measure space X. Although, as the author remarks, the concept is a special case of the direct sums treated by the reviewer [same J. 8, 386–392 (1941); MR 3, 50], the present treatment has the advantage of describing a canonical procedure for forming the domain of the measure in the direct union. (In the more general theory one is forced to speak of "a" direct sum; here one can speak of "the" direct union.) The (unavoidable) disadvantage of the treatment is that the domain it describes is rather small. In addition to proving the Fubinoid theorems to be expected in this field, the author discusses several special examples and shows that his theory yields certain extensions of the usual measures in product spaces.

P. R. Halmos (Chicago, Ill.).

Nicolescu, Miron. Contributions à une analyse du plan, de type hyperbolique. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 7–51 (1952). (Romanian. Russian and French summaries)

Let $u(x, y)$ be a real function defined in the interval I: $a \leqq x \leqq \alpha$, $b \leqq y \leqq \beta$, and let
$\Delta_2 u(x, y) = u(v+h, y+k) - u(x+h, y) - u(x, y+k) + u(x, y)$.
Then the author defines the H-derivative as

$$Du(x, y) = \lim_{h \to 0, k \to 0} \Delta_2 u(x, y)/hk,$$

whenever the limit exists. For smooth $u(x, y)$, the H-derivative is u_{xy}. The function $u(x, y)$ is H-continuous if $\Delta_2 u \to 0$ as $h, k \to 0$.

In this note, the author develops some of the analogues of classical results concerning derivatives, and their integrals. For example, he establishes a formula for integration by parts and he solves the differential equation $D^{n+1}u(x, y) = D(D^n u(x, y)) = 0$ for $n = 1, 2, \cdots$ and finds $u(x, y) = \sum_{k=0}^{n} [x^k g_k(y) + y^k f_k(x)]$, where g_0 and f_0 are arbitrary and g_k and f_k are k times differentiable, as the most general solution to the differential equation. These latter

functions are the pseudo-polynomials of Popoviciu [Mathematica, Cluj 8, 1–85 (1934)] and Marchand [J. Math. Pures Appl. (9) 6, 337–425 (1927)]. The present author continues by studying repeated and generalized H-derivatives and then by obtaining the following analogue of a result due to Schwarz. Theorem: If the generalized second H-derivative vanishes identically in I, i.e., if

$$\lim_{h\to 0,k\to 0} \Delta_2^2 u(x,y)/h^2k^2 \equiv 0$$

in I, then $u(x,y)\equiv xg_1(y)+yf_1(x)+g_0(y)+f_0(x)$, where f_k, g_k are continuous but arbitrary. These last functions $u(x,y)$ are called H-linéaire. The author then shows that for each continuous $u(x,y)$ in I, there exists exactly one H-linear function coinciding with $u(x,y)$ on the frontier of I.

The author then considers sequences of pseudo-polynomials and shows that each H-continuous function $u(x,y)$ can be uniformly approximated on I by an interesting sequence of pseudo-polynomials. He also considers H-entire series, i.e., series of the form $\sum_0^\infty [\varphi_k(x)+\psi_k(y)]x^k y^k$ introduced by Cioranescu [Enseignement Math. 34, 220–227 (1936); Mathematica, Cluj 14, 180–189 (1938)] and finds a domain of convergence. These considerations lead, naturally, to an analogue of Taylor's theorem with a remainder, and then to expansion theorems for functions infinitely H-derivable and H-analytic.

The author concludes by considering totally monotone H-functions and by obtaining the following analogue of a result due to S. Bernstein. If $u(x,y)$ is absolutely totally monotone (in the present author's sense) in I, then $u(x,y)$ and all its H-derivatives are H-analytic in I.

M. Reade (Paris).

Coppel, W. A. The solution of equations by iteration. Proc. Cambridge Philos. Soc. 51, 41–43 (1955).

The author examines continuous functions $f(x)$ in intervals which they "contract" $[a\leqq x\leqq b,\ a\leqq f(x)\leqq b]$ from the point of view of iteration processes: $x_{n+1}=f(x_n)$ $(n=1,2,\cdots)$. He proves that these sequences converge for any x_1 [of course, to a root of $x=f(x)$] if and only if the equation $f_2(x)\equiv f(f(x))=x$ has no roots besides the roots of the equation $f_1(x)\equiv f(x)=x$. (The main tool in the proof is a "preliminary theorem" asserting, that under these conditions for any $c\in(a,b)$, $f_n(c)\gtrless c$ either for every $n\geqq 1$ or for none, where as usual $f_{n+1}(x)=f[f_n(x)]$.) Moreover, $f(x)=x$ has exactly one root $x=r$ to which every iteration sequence converges in a steadily improving manner

$$[|r-f_{n+1}(x)|<|r-f_n(x)|,\ n=1,2,\cdots]$$

if and only if $|r-f(x)|<|r-x|$ for every $x\neq r$. This shows that here weaker conditions than the usual ones are made possible by the preliminary supposition $[f(x)\in(a,b)$ continuous for $x\in(a,b)]$, which however may not be fulfilled in every practically occurring case. *J. Aczél.*

Michael, J. H. An approximation to a rectifiable plane curve. J. London Math. Soc. 30, 1–11 (1955).

Let C be a continuous closed curve not necessarily simple of the xy-plane E_2 of finite Jordan length; let $O'(C)\subset E$ be the open set (inside of C) of all points $(x,y)\in E_2$ having a non-zero order $u(x,y;C)$ with respect to C, and $K(C)$ the set of points (x,y) covered by C in E_2. I. If $f(z)$, $z=x+iy$, is holomorphic in $O'(C)$ and continuous in $O'(C)+K(C)$, then $(C)\int f(z)dz=0$. II. If $f(x,y)$ is continuous on $O'(C)+K(C)$, if $f_x(x,y)$ exists everywhere in $O'(C)$ except on a set F which is the finite sum of sets having each finite linear measure, if the product $u(x,y;C)f_x(x,y)$ is L-integrable in $O'(C)$, then

$$O'(C)\int\int u(x,y;C)f_x(x,y)dxdy=(C)\int f(x,y)dy.$$

Thus both theorems (Cauchy, and Gauss theorems) are given under conditions lighter than usual. Both theorems are proved by a new process of approximation of C by means of polygonal paths which are sums of perimeters of squares.

L. Cesari (Lafayette, Ind.).

***Rinow, Willi. Der Begriff der Ergiebigkeit eines Vektorfeldes und der Gaussche Integralsatz.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 284–289. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Let $\mathfrak{v}$ be a vector field in 3-dimensional Euclidean space. The author defines the function $\eta(B;\mathfrak{v})=\int_{R(B)}\mathfrak{v}\mathfrak{n}\,d\sigma$ firstly for an interval B and then uses known extension theorems to define it for all numbers of a certain Borel σ-ring, where $R(B)$ is the frontier of B, $\mathfrak{n}$ the outward normal and σ an appropriate measure function. A representation of $\eta(B;\mathfrak{v})$ is obtained from the Radon-Nikodym theorem and this leads to a simple proof of Gauss' theorem. Generalizations are given which include Green's formula. Full details are to appear in a later paper. *H. G. Eggleston.*

Theory of Functions of Complex Variables

***Richard, Ubaldo. Teoria delle funzioni.** Casa Editrice Libraria Rosenberg & Sellier, Torino, undated. vii+367 pp. 2800 lire.

An unusually careful introduction to the elements of real and complex variables. Contents: Ch. I. Rings, fields, ordered fields. Ch. II. $[n]$-space, elements of set theory, metric spaces, Peano-Jordan measure, Borel measure, Lebesgue measure. Ch. III. Point functions and set functions, Jordan curves, arc length, Jordan curve theorem. Convex regions. Convex functions. Ch. IV. Complex numbers, series of complex numbers. Power series. Ch. V. Riemann integral. Ch. VI. Lebesgue integral. Ch. VII. Riemann-Stieltjes integral. Ch. VIII. Complex variable. Regular functions, Cauchy's theorem, calculus of residues, analytic continuation, Laurent series. Ch. IX. Harmonic functions.

The presentation is concise, but clear. The material is restricted to the basic facts with few illustrative applications. There are some nice individual touches in the choice of these illustrations, e.g. the example of an infinitely differentiable function of x which is not regular at any point of the x-axis. *W. H. J. Fuchs* (Ithaca, N. Y.).

Combes, Jean. Sur les zéros des dérivées successives des fonctions analytiques. I, II. C. R. Acad. Sci. Paris 240, 39–41, 145–146 (1955).

The author is concerned with the Abel-Gončarov uniqueness problem: one seeks conditions on an analytic function f and a sequence $\{z_n\}$ such that $f^{(n)}(z_n)=0$ $(n=0,1,2,\cdots)$ shall imply $f(z)\equiv 0$. Most of the results which the author announces are already in the literature. [See, e.g., Gončarov, Ann. Sci. Ecole Norm. Sup. (3) 47, 1–78 (1930); Kakeya, Proc. Phys.-Math. Soc. Japan 14, 125–138 (1932); Ibragimov, Izv. Akad. Nauk SSSR. Ser. Mat. 1939, 553–568; MR 1, 310; Džrbašyan, ibid. 16, 225–252 (1952); MR 14, 33.] *R. C. Buck* (Madison, Wis.).

Rudin, Walter. Multiplicative groups of analytic functions. Proc. Amer. Math. Soc. 6, 83–87 (1955).

The connectivity $k(D)$ of a proper subdomain D of the Riemann sphere is the number of components of the complement of D, if finite, and ∞ otherwise. Let $M(D)$ denote the group of all regular single-valued analytic functions on D without zeros on D. The author shows that if D_1, D_2 are proper subdomains of the Riemann sphere, then $M(D_1)$ and $M(D_2)$ are isomorphic if and only if $k(D_1)=k(D_2)$.

M. Henriksen (Lafayette, Ind.).

Denjoy, Arnaud. Points critiques logarithmiques des inverses des fonctions analytiques uniformes. C. R. Acad. Sci. Paris **240**, 22–25 (1955).

This paper concerns the nature of the singularities of the inverse function of a uniform function whose singularities form a set which is totally disconnected. The author begins by defining an accessible singularity a of an analytic function $z(x)$ as one which can be reached by analytic continuation in a regular branch $\zeta(x)$, starting from any regular point of $z(x)$. Leaving aside poles, a is an essential point if, in some circle containing a, $\zeta(x)$ is uniform but not meromorphic. It is pointed out that the set of essential points may have limiting points which are inaccessible singularities [see Denjoy, Rend. Circ. Mat. Palermo **50**, 1–95 (1926)]. Now suppose that although $\zeta(x)$ is multiform, $f(x)=\zeta[(x-a)^p]$ is uniform in some circle centre a, where $p\geqq 2$ is an integer. If $f(x)$ is meromorphic at a, then a is an algebraic point of $z(x)$, while if a is essential for $f(x)$ then a is a critical essential point for $z(x)$. Finally, a is called a logarithmic point of $z(x)$ if the prolongation of $\zeta(x)$ has an infinity of branches in the neighbourhood of a. An accessible singularity which is neither polar nor algebraic is called transcendental. Using these definitions the following theorem is proved: Let $x=F(z)$ be uniform and the closure E of its essential points be totally disconnected. Then the singular transcendental points a of $z(x)$ are the different unique limiting values to which $F(z)$ may tend as z describes a simple arc containing no points of E but terminating at a point b of E. In general, save in very special cases, a is a logarithmic point. In particular, if $x=F(z)$ is meromorphic, then every transcendental point of $z(x)$ is logarithmic. [See also Denjoy, C. R. Acad. Sci. Paris **145**, 106–108 (1907).] If a is polar or algebraic for $\zeta(x)$ then $a=F(b)$ and $F(z)$ being regular or having a pole at b, it follows that b does not belong to E. The long and difficult proof that a is not an essential point is the crux of the paper.

R. Wilson (Swansea).

Badalyan, G. V. Generalization of Taylor's series and some questions of the theory of analytic and quasi-analytic functions. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Nauk **6**, no. 5–6, 1–63 (1953); **7**, no. 1, 3–33 (1954). (Russian. Armenian summary)

The author continues his investigations on generalized factorial series and their associated generalized power series [Akad. Nauk Armyan. SSR. Soobšč. Inst. Mat. Meh. **5**, 13–84 (1950); MR **15**, 946]. If $f(z)$ is represented by a generalized factorial series based on $\{\gamma_\nu\}$ and $\gamma_\nu'\geqq\gamma_\nu$, $\sum 1/\gamma_\nu'=\infty$, then $f(z)$ is also represented by its factorial series based on $\{\gamma_\nu'\}$, in at least as large a domain. Expansion of a function $f(z)$ in a generalized factorial series is equivalent to having $f(z)=\int_0^1 t^{z-1}\phi(t)\,dt$ with $\phi(t)$ given by a generalized Taylor series. Estimates for the derivatives of a function which has a generalized Taylor series lead the author to consider quasi-analytic classes $C_{\{m_n\}}$ with the special property that $n^{-1}m_n/m_{n-1}$ increases. The author's main theorem then states that if $\phi(x)$ belongs to such a class on $[0, 2]$, then for any u, $0<u<2$, $\phi(x)$ is represented by its generalized Taylor series on $(0, u]$, based on the sequence $\gamma_\nu=\delta m_\nu/m_{\nu-1}$, provided that δ is sufficiently large.

The following applications are given. (1) The series $\sum_{n=1}^{\infty} c_n/z^n$ is formally equivalent to a generalized factorial series $\sum_{n=0}^{\infty} a_n/\prod_{\nu=1}^{n}(z-\gamma_\nu)$; the sum of the latter (convergent) series can sometimes be taken as a sum for the former series. The author gives some sufficient conditions for a series to be summable in this way (to sum a numerical series take $z=1$) and shows that the method is regular and that if a series is summable with two sequences $\{\gamma_\nu\}$, the sums are the same. (2) The representation of a quasi-analytic function by a generalized factorial series allows one to obtain a quasi-analytic extension of a function determined by the values of its successive derivatives at a point. (3) Under certain conditions the regions of convergence of the generalized factorial series $\sum a_n/\prod_{\nu=1}^{n}(1+z/\beta_\nu)$ and the Dirichlet series $\sum a_n \exp\{-z\sum_{\nu=0}^{n} 1/\beta_\nu\}$ coincide. This allows one to give an analytic extension of the function defined by a Dirichlet series by means of another Dirichlet series built on a sequence $\{\beta_\nu'\}$ (using the theorem quoted at the beginning of this review). There are similar results for generalized Newton series $\sum a_n\prod_{\nu=1}^{n}(1-z/\alpha_\nu)$.

R. P. Boas, Jr. (Evanston, Ill.).

***Berg, Lothar. Abschätzungen von Potenzreihenteilsummen.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 193–196. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

This is a short description of an expository nature of previously reviewed results of the author [Math. Nachr. **11**, 213–218 (1954); MR **16**, 24]. *F. Herzog.*

Roux, Delfina. Lacune unilaterali, emisimmetria di tratti e teorema di Fabry. Boll. Un. Mat. Ital. (3) **9**, 399–408 (1954).

Let $\sum a_n z^n$ be a power series of radius of convergence 1. The author proves two theorems giving sufficient conditions for the point $z=1$ to be a singular point. The first of these theorems gives the following conditions: Let there exist a sequence of indices $n_h\to+\infty$, a sequence of numbers of unit modulus ω_h, and a real number θ, $0<\theta<1$, with the properties: (a) $\Re(a_{n_h}\omega_h)>0$; (b) $[\Re(a_{n_h}\omega_h)]^{1/n_h}\to 1$; and (c) as the integer m ranges through the interval $n_h(1-\theta)\leqq m\leqq n_h$, the number of variations in sign of $\Re(a_m\omega_h)$ is $o(n_h)$. Under these hypotheses $z=1$ is a singular point. The "left" intervals $[n_h(1-\theta), n_h]$ can, of course, be replaced by "right" intervals $[n_h, n_h(1+\theta)]$. In the second theorem the conditions involve two sequences of indices $\{p_h\}$ and $\{q_h\}$ with $p_h<q_h<(1+\theta)p_h$ with the properties $\Re((a_{p_h+k}+a_{q_h-k})\omega_h)>0$ for $k=0$ and $\geqq 0$ for $k=1, 2, \cdots, [(q_h-p_h)/2]$ and

$$[\Re((a_{p_h}+a_{q_h})\omega_h)]^{1/p_h}\to 1.$$

F. Herzog (E. Lansing, Mich.).

Herzog, Fritz. A note on power series which diverge everywhere on the unit circle. Michigan Math. J. **2** (1953–54), 175–177 (1955).

The author constructs a power series $\sum_0^\infty a_n z^n$, and a cosine series $\sum_0^\infty a_n \cos n\theta$ with $a_n\geqq 0$ and $\lim a_n=0$ such that (with $z=e^{i\theta}$) both diverge for all θ. *R. C. Buck.*

Erdös, P., and Macintyre, A. J. Integral functions with gap power series. Proc. Edinburgh Math. Soc. (2) **10**, 62–70 (1954).

Let the entire function $f(z)$ have the Taylor series

$$(1) \qquad \sum_{n=0}^{\infty} a_n z^n \quad (\lambda_{n+1}>\lambda_n),$$

and let

$$M(r)=\max_{|z|=r}|f(z)|, \quad m(r)=\min_{|z|=r}|f(z)|,$$

$$\mu(r)=\max_{n=0,1,\cdots}|a_n|r^n.$$

The authors deal with conditions on the gaps in (1) which imply that

$$(2) \qquad \limsup_{r\to\infty} m(r)/M(r)=\limsup_{r\to\infty}\mu(r)/M(r)=1.$$

Their first result: the condition (3) $\sum(\lambda_{n+1}-\lambda_n)^{-1}<\infty$ implies (2); conversely, if the sequence $\{\lambda_n\}$ does not satisfy (3), there exists an entire function of the form (1) with

$$\limsup \mu(r)/M(r)\leqq\tfrac{1}{2}, \quad \text{and} \quad \limsup m(r)/M(r)\leqq\tfrac{1}{2}.$$

If (3) is replaced by the weaker condition that (4) $\sum(\lambda_{n+h}-\lambda_n)^{-1}<\infty$ for some positive h, then

$$\limsup \mu(r)/M(r)\geqq(2h-1)^{-1};$$

but even with $h=2$, (4) does not imply that

$$\limsup m(r)/M(r)>0.$$

On the other hand, if (4) is false for every h, there exists a series (1) such that $\lim \mu(r)/M(r)=\lim m(r)/M(r)=0$. Finally, if $\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)^{-1}=o(\log\lambda_n)$ and $f(z)$ is of finite order, or if $\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)^{-1}=O(\log\lambda_n)$ and $f(z)$ is of order zero, then (2) holds. *G. Piranian* (Ann Arbor, Mich.).

Clunie, James. The asymptotic paths of integral functions of infinite order. J. London Math. Soc. **30**, 80–83 (1955).

The author formulates Milloux's theorem [Compositio Math. **1**, 305–313 (1935)] in the following way. Let $f(z)$ be entire, of infinite order, and let $\eta>0$ be given. In the annulus $R^{1/(1+\eta)}=r\leqq|z|\leqq R$ let there be n paths L_i joining the bounding circles for which there are corresponding numbers a_i such that $|f(z)-a_i|<\epsilon$ on L_i, while $|a_i-a_j|>\epsilon^{1/\delta}$. When $\epsilon<K\{M(r,f)\}^{-1+\delta}$, $\delta>0$, then for all sufficiently large R,

$$n\leqq 2(1+\eta)\frac{\log\log M(R,f)}{\log R}.$$

The proof starts out like Macintyre's proof [J. London Math. Soc. **10**, 34–39 (1935)] of the Denjoy-Ahlfors theorem on asymptotic paths of entire functions of finite order.

J. Korevaar (Madison, Wis.).

Garabedian, P. R., and Schiffer, M. A coefficient inequality for schlicht functions. Ann. of Math. (2) **61**, 116–136 (1955).

Let S denote the class of functions of the form

$$w=f(z)=z+b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\frac{b_3}{z^3}+\cdots,$$

univalent and regular in $|z|>1$. The inequality $|b_1|\leqq 1$ is due to Bieberbach (S.-B. Preuss. Akad. Wiss. **1916**, 940–955], $|b_2|\leqq 2/3$ is due to Golusin [Mat. Sb. N.S. **3(45)**, 321–330 (1938)] and Schiffer [Bull. Soc. Math. France **66**, 48–55 (1938)]. The authors now prove that $|b_3|\leqq\frac{1}{2}+e^{-6}$. All these estimates are sharp.

The crux of the argument consists in establishing the differential equation for the functions of S which extremalize Re b_3 in the form

$$\left(\frac{dw}{dz}\right)^2(w^2-b_1)=\left(z^2-\frac{b_1}{z}-\frac{1}{z^2}\right)^2.$$

They show that this equation has two solutions in S, one of which, $w=(z^4+2+z^{-4})^{1/4}$, gives $b_3=\frac{1}{2}$, and the other of which satisfies

$$w(w^2-4ie^{-3})^{1/2}-4ie^{-3}\log\frac{w+(w^2-4ie^{-3})^{1/2}}{2z}=z^2+\frac{1}{z^2}+6ie^{-3}$$

and gives $b_3=\frac{1}{2}+e^{-6}$. If $f(z)\in S$ has real coefficients, the range of b_3 is shown to be the interval $[-\frac{1}{2}-e^{-6},\frac{1}{2}]$.

A number of consequences are obtained, such as

$$\text{Re}\,\{b_3-3ib_1\}\leqq 3, \quad \text{Re}\,\{b_3+2b_1\}\leqq 2,$$

and simpler related problems are also solved.

W. K. Hayman (Exeter).

Reade, Maxwell. Sur une classe de fonctions univalentes. C. R. Acad. Sci. Paris **239**, 1758–1759 (1954).

For the class of close-to-convex schlicht functions in $|z|<1$,

$$f(z)=a_1z+a_2z^2+\cdots+a_nz^n+\cdots$$

introduced by W. Kaplan [Michigan Math. J. **1**, 169–185 (1953); MR **14**, 966] and which are characterized by the property that there exists a function $\phi(z)$ analytic, schlicht and convex in $|z|<1$ for which $\Re[f'(z)/\phi'(z)]>0$, $|z|<1$, the author points out that the inequalities

$$(1) \qquad |a_n|\leqq n|a_1|, \quad n=2,3,\cdots,$$

follow easily from well known inequalities for the coefficients of the power series of functions of positive real part in $|z|<1$, and of the convex functions $\phi(z)$. The class of close-to-convex functions contains the star-like functions, the functions convex in one direction introduced by the reviewer [Amer. J. Math. **58**, 465–472 (1936)] and the functions for which the rotation of the boundary of the map of the unit circle is bounded by 4π [see V. Paatero, Ann. Acad. Sci. Fenn. Ser. A. **37**, no. 9 (1933)]. For these sub-classes the inequalities (1) are already known, but the author's new proof handles all these special cases at once in a simple way and extends (1) to a wider class than was previously considered. *M. S. Robertson.*

Goodman, A. W., and Reich, E. On regions omitted by univalent functions. II. Canad. J. Math. **7**, 83–88 (1955).

[For part I see Goodman, Bull. Amer. Math. Soc. **55**, 363–369 (1949); MR **10**, 601.] Let $w=f(z)=z+a_2z^2+\cdots$ be univalent in $|z|<1$ with image D_f. Let $1/4\leqq s\leqq 1$, $s\leqq r\leqq s/2s^{1/2}-1$, and let $L(r,s)$ denote the length of arc of the circle $|w|=r$ not covered by D_f. If $f(z)$ omits the value s, then a sharp upper bound of $L(r,s)$ for these r is given. For every $f(z)$ of the above univalent class, a lower bound for the area A_f of the intersection of D_f with $|w|<1$ is also obtained [cf. Jenkins, Amer. J. Math. **75**, 406–408 (1953); MR **14**, 967]. *W. W. Rogosinski.*

Jenkins, J. A. On explicit bounds in Schottky's theorem. Canad. J. Math. **7**, 76–82 (1955).

Let S be the class of functions $f(z)=a_0+a_1z+\cdots$ regular in $|z|<1$ and such that $f(z)\neq 0$ or 1. Schottky's Theorem states that the least upper bound $K(a_0,r)$ for $|f(re^{i\theta})|$,

when $f \in S$ and a_0 is given, is finite. R. M. Robinson [Bull. Amer. Math. Soc. 45, 907–910 (1939); MR 1, 112] showed that

$$K(a_0, r) \leqq \tfrac{1}{16}\{[16|a_0|+18]^{(1+r)/(1-r)}+18\}$$

and if $16|a_0|-18>0$, also

$$K(a_0, r) > \tfrac{1}{16}\{[16|a_0|-18]^{(1+r)/(1-r)}-18\}.$$

The reviewer [Proc. Cambridge Philos. Soc. 43, 442–454 (1947); MR 9, 84] obtained

$$K(a_0, r) < \{\max(1, |a_0|)e^r\}^{(1+r)/(1-r)}, \quad K(-1, r) > e^{2\pi r/(1-r)}.$$

The author obtains very simply the sharper upper bound

$$K(a_0, r) < \tfrac{1}{16}\beta^{(1+r)/(1-r)}+1,$$

where $\beta \leqq e^r \max(1, |a_0|)$, and $\beta \leqq 16|a_0|+8$. Here 8 cannot be replaced by less than $e^r-16=7.14\cdots$. He then sharpens the reviewer's bound $|a_1| \leqq 2|a_0|\{|\log|a_0||+5\pi\}$ replacing 5π by 7.77. The best possible constant is at least $\Gamma^4(\frac{1}{4})/4\pi^2=4.376\cdots$ when $a_0=-1$. [This follows from J. E. Littlewood, Lectures on the theory of functions, Oxford, 1944, Theorem 228, p. 196; MR 6, 261.]

W. K. Hayman (Exeter).

Walsh, J. L., and Davis, Philip. Interpolation and orthonormal systems. J. Analyse Math. 2, 1–28 (1952). (Hebrew summary)

Let a region B contain the origin in its interior and be bounded by a finite number of mutually disjoint Jordan curves in the z $(=x+iy)$ plane; let (f, g) be the inner product $\iint_B f(z)\overline{g(z)}\,dxdy$, $L^2(B)$ the class of functions $f(z)$ which are regular in B, with $(f, f)<\infty$. As Bergman has shown, there exists an infinity of orthonormal sequences $\{\varphi_n^*(z)\}$ (i.e. $(\varphi_i^*, \varphi_k^*)=\delta_{ik}$), any of them closed in $L^2(B)$. Every $f \in L^2(B)$ possesses a generalised Fourier expansion $f(z)=\sum_0^\infty a_n\varphi_n^*(z)$, with $a_n=(f, \varphi_n^*)$, which is uniformly and absolutely convergent in any closed subregion of B. The series may, however, be defined and converge for a class S larger than B; and the a_n need not be defined as above, but as linear functionals $L_n^*(f)$ of $f(z)$. In the trivial case when B is the interior of the unit circle we may choose $L_n(f)=(f, g_n)$, $g_n(z)=z^n$. Then $L_n^*(f)=c_nL_n(f)$, $\varphi_n^*=c_nz^n$, where $c_n=(n+1)^{1/2}\pi^{-1/2}$; $a_n=L_n^*(f)=f^{n}(0)/(n!c_n)$; the Fourier expansion reduces to the Taylor series; again $L_n^*(f)=(f, \varphi_n^*)$ if $f \in L^2(B)$.

In §2, the convergence is discussed of orthonormal interpolatory systems which arise from the selection $L_n(f)=f(\alpha_n)$, $\alpha_n \to 0$ $(n \to \infty;\ \alpha_n \in B)$; each L_n^* is a linear combination of $L_0, L_1, \cdots, L_n$; $f(z)$ is required to be regular in some region containing the α_n's and 0. The main results are deduced in §3. Given B and a set $\{L_n\}$ of linear functionals each of which is defined for any f of $L^2(B)$ or of a wider class S, such that each L_nf is bounded over $L^2(B)$ and that the set is independent; then there exist linear combinations L_n^* of the L_n, $L_n^*=\sum_{j=0}^n c_{jn}L_j$, and functions $\varphi_n^* \in L^2(B)$ such that (i) $(\varphi_i^*, \varphi_k^*)=\delta_{ik}$ and (ii) $L_i^*(\varphi_k^*)=\delta_{ik}$ $(i, k=0, 1, 2, \cdots)$. If $f \in L^2(B)$, then $L_n^*f=(f, \varphi_n^*)$. The doubly orthogonal set $\{\varphi_n^*\}$ is complete in $L^2(B)$ if, and only if, $\{L_n\}$ is complete, i.e. if $L_nf=0$ $(n=0, 1, 2, \cdots;\ f \in L^2(B))$ implies that $f(z)=0$. Explicit representations of the $\varphi_n^*(z)$ are given by means of the Bergman kernel function for the region B or by Green's function for B. When $\{L_n\}$ is defined for $S \supset L^2(B)$ and $\{L_n\}$ is complete in S, then a function $f(z)$ of S is $L^2(B)$ if and only if $\sum_0^\infty |L_n^*(f)|^2<\infty$.

The Bergman "doubly orthogonal" functions are shown to be included in the present results. The latter are further applied to derive an inequality which is in the Schwarz circle of ideas, to the general problem of interpolation in an arbitrary region, and finally to a problem of best approximation to functions regular in a subregion of B by functions of $L^2(B)$, which is related to previous work by Walsh and Nilson, and by Davis.

H. Kober (Birmingham).

Davis, Philip, and Pollak, Henry. Linear functionals and analytic continuation problems. Pacific J. Math. 3, 47–72 (1953).

Results of the paper reviewed above are applied to various problems on analytic continuation. The significance of the generalised Fourier expansion $f(z)=\sum_0^\infty L_n^*(f)\varphi_n^*(z)$, where $L_n^*f=(f, \varphi_n^*)$, lies in the fact that the L_n^* may be defined over a wider class S than is the inner product. Thus, given B and the L_n, criteria are obtained for analytic continuation and are illustrated by examples in §3; also the problem corresponding to Schur's Theorem on the Taylor coefficients of a function bounded in the unit circle is discussed. In §4, a change-of-sign theorem is treated; it corresponds to the Fatou-Pólya theorem: let B be a region subject to certain conditions, let the L_n be independent and each of them bounded over $L^2(B)$, and let $\{L_n\}$ be complete in $S \supset L^2(B)$. If $f(z)$ is regular in B, but is not $L^2(B)$ (e.g. $f(z)$ is only $L^1(B)$) and possesses the Fourier coefficients $a_n=L_n^*(f)$ then there are signs $\epsilon_n=\pm 1$ such that, if $a_n^*=\epsilon_na_n$ are the coefficients of a $f^*(z)$ which is regular in B, then $f^*(z)$ can not be continued analytically across any portion of the boundary of B. It is an open question whether the $f^*(z)$ exists in the general case. Certainly it does for special systems $\{L_n(f)\}$, provided that B is bounded by a finite number of Jordan curves. In §5 a gap theorem is deduced. Let the set $\{L_n\}$ be defined over $S \supset L^2(B)$, complete with respect to S, and let $L_n^*(f)=(f, p_n)$ where $\{p_n\}$ is a complete orthonormal set for $L^2(B)$ and is, in addition, regular in some closed domain containing B in its interior. If $|a_n| \geqq \alpha$ $(n=0, 1, 2, \cdots;\ \alpha>0)$ then there exists a sequence $\{\delta_n\}$, $\delta_n=0$ or 1, such that, if δ_na_n are the Fourier coefficients of a function $f^*(z)$ which is regular in B, then $f^*(z)$ possesses the boundary of B as natural boundary. The existence of $f^*(z)$ is ensured by similar conditions as in the previous theorem.

H. Kober (Birmingham).

Mergelyan, S. N., and Džrbašyan, M. M. On best approximations by rational functions. Dokl. Akad. Nauk SSSR (N.S.) 99, 673–675 (1954). (Russian)

The authors prove a direct and a converse theorem on approximation on the unit circumference by rational functions with prescribed poles. Let $\{\lambda\}$ be a sequence of complex numbers and let $\epsilon(\lambda_1, \cdots, \lambda_n)$ denote

$$\{1+\textstyle\sum'(1-|\lambda_k|)\}^{-1}+\{1+\sum''(1-|\lambda_k|^{-1})\}^{-1} \quad (k \leqq n),$$

with $\sum'$ containing those λ_k for which $|\lambda_k|<1$, and $\sum''$ containing the rest. If $f(z)$ has a pth derivative in Lip α, then (provided no λ has absolute value 1) the best approximation to f by rational functions of degree n with poles $\lambda_1, \cdots, \lambda_n$ is at most a constant multiple of

$$\{\epsilon(\lambda_1, \cdots, \lambda_n)|\log \epsilon(\lambda_1, \cdots, \lambda_n)|\}^{p+\alpha}.$$

If one of the series in the definition of ϵ diverges (when extended over all λ_k), if for every set of n elements of $\{\lambda\}$ the best approximation to f is at most $\{\epsilon(\lambda_1, \cdots, \lambda_n)\}^{p+\alpha}$, and if the arc γ of $|z|=1$ contains no limit point of $\{\lambda\}$, then f has a pth derivative in Lip α on γ.

R. P. Boas, Jr.

Evgrafov, M. A. Properties of complete systems in spaces of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **98**, 717–718 (1954). (Russian)

Let D be a domain in the z-plane, $A(D)$ the space of functions regular in D with the topology of uniform convergence in closed $E \subset D$. If F is a closed set, let $A(F)$ be the space of functions regular in F with the topology of uniform convergence in a set E strictly containing F. If $\phi(z)=\sum_{n=0}^{\infty} \alpha_n z^n$ ($\alpha_n \neq 0$) is an entire function, let $A(\phi, D)$ be the space of functions

$$F(z)=(1/2\pi i)\int_L \phi(z\xi) f(\xi)\, d\xi, \quad f \in A(CD),$$

L a path encircling D. The topology in $A(\phi, D)$ is defined by: $F_n \to F$ means $f_n \to f$ in $A(CD)$. The space $A(\phi, \bar{D})$ ($\bar{D}$ closure of D) is similarly defined. The author mentions that weak convergence and convergence in the topology coincide in $A(D)$ and $A(\phi, D)$. Assume now that D and its complement are simply connected and that $0 \in D$. Let $\phi_n(z)=\sum_{k=0}^{\infty} \gamma_{nk} z^k \in A(\phi, D)$ ($n=0, 1, \cdots$) and let $(\beta_{nk})_{n,k=0}^{\infty}$ be the transpose of an inverse of the matrix (γ_{nk}). Let $\psi_n(\xi)=\sum_{k=0}^{\infty} \beta_{nk}\alpha_k \xi^k$. The author states the following theorem. Let $D_1 \subset D$. Every $F(z) \in A(\phi, D)$ can be expanded in a series $F(z)=\sum_{n=0}^{\infty} c_n \phi_n(z)$ (in $A(\phi, D_1)$) if and only if $\phi(z\xi)=\sum_{n=0}^{\infty} \phi_n(z)\psi_n(\xi)$ in $A(\phi, \bar{D}_1)$; uniformly for $\xi \in \bar{D}_2 \subset D$ and also in $A(\phi, \bar{D})$, uniformly for $z \in \bar{D}_2 \subset D_1$.

W. H. J. Fuchs (Ithaca, N. Y.).

Walsh, Joseph L. Sur la représentation conforme des aires multiplement connexes. C. R. Acad. Sci. Paris **239**, 1572–1574 (1954).

L'A. prouve que tout domaine plan, limité par un nombre fini de courbes de Jordan disjointes, peut être représenté conformément sur un domaine Δ défini par des inégalités de la forme

$$A_1 < |\prod (z-a_i)^{M_i} \prod (z-b_j)^{-N_j}| < A_2$$

les exposants M_i et N_j pouvant être irrationnels. La méthode utilise l'approximation de la mesure harmonique par des potentiels de masses discrètes. *J. Lelong* (Lille).

Jacobsthal, Ernst. Über die Kreise, die durch eine gegebene lineare Funktion auf einen konzentrischen Kreis abgebildet werden. Norske Vid. Selsk. Skr., Trondheim **1953**, no. 3, 22 pp. (1954).

The author considers the problem of finding the circles $|z-\mu|=r$ which are transformed into concentric circles by the linear fractional transformation

$$(1) \qquad w=\frac{az+b}{cz+d}, \quad D=ad-bc \neq 0, \quad c \neq 0.$$

He shows that all such centers μ lie on a hyperbola through the points $\alpha=-d/c$, $\beta=a/c$, and the two fixed points z_1 and z_2 of (1). The center of the hyperbola is at

$$\vartheta=\tfrac{1}{2}(z_1+z_2)=\tfrac{1}{2}(\alpha+\beta)$$

and the asymptotes are perpendicular. The hyperbola degenerates into its asymptotes if and only if $Q=(a+d)^2/D$ is real. If r is sufficiently large and prescribed, there are exactly five circles with radius r and concentric images. He then shows that one may remove two segments from the hyperbola between the points α, β, z_1, and z_2 such that each part of what remains is the center of exactly one circle with concentric image. *G. Springer* (Muenster).

Kuroda, Tadashi. Theorems of the Phragmén-Lindelöf type on an open Riemann surface. Osaka Math. J. **6**, 231–241 (1954).

Given an open Riemann surface F, consider its exhaustion by nested subregions F_n with analytic boundaries Γ_n. Let u_n be the harmonic function on the set $D_n=F_n-F_{n-1}-\Gamma_{n-1}$, determined by constant boundary values 0 and $\log \sigma_n$ on Γ_{n-1} and Γ_n respectively. If the constant σ_n is so chosen that the conjugate period $\int_{\Gamma_n} dv_n = 2\pi$, then the function $\exp(u_n+iv_n)$ with suitable additive constants of v_n maps D_n onto the annulus $1<|z|<\sigma_n$ less some radial slits, and σ_n appears as the modulus of D_n [Sario, C. R. Acad. Sci. Paris **230**, 269–271 (1950); Duke Math. J. **20**, 279–286 (1953); MR **11**, 342; **14**, 969]. It is known that F is parabolic if and only if the modulus $\sigma=\sum_1^{\infty} \sigma_n$ of $F-F_0$ is infinite [Sario, loc. cit., and Noshiro, Nagoya Math. J. **3**, 73–79 (1951); MR **13**, 833]. The functions u_n+iv_n on D_n map $F-F_0$ onto a finite or infinite strip $\bar{P}$: $0<u_n<\log \sigma$, $0<v_n<2\pi$ minus certain vertical and horizontal slits [Noshiro, loc. cit.].

The author proves the following theorems: (1) Let G be a suitably restricted noncompact subregion of F, and $\theta(r)$ the length of the intersection θ_r of $u=r$ ($0<r<\log \sigma$) with the image $\bar{G} \subset \bar{F}$ of G. Then G can be conformally imbedded into a parabolic Riemann surface if and only if $\int^{\log \sigma} 1/\theta(r)\, dr$ diverges for some $\{F_n\}$. (2) Consider an analytic function $f(p)$ with a single-valued modulus on G. Suppose $|f(p)| \leq 1$ on the relative boundary C of G, $|f(p_0)|>1$ for some $p_0 \in G$, and set $r_0=u(p_0)$, $M(r)=\max |f(p)|$ on θ_r. Then

$$\liminf_{r \to R} (\log M(r))^2 \Big/ \int_{r_0}^{r} \frac{dr}{\theta(r)} > 0.$$

(3) Assume $f(p)$ is single-valued analytic on G with $\operatorname{Re}\{f(p)\}=0$ on C, and let $M^*(r)=\max |\operatorname{Re}\{f(p)\}|$ on θ_r. If

$$\liminf_{r \to R} (M^*(r))^2 \Big/ \int_{r_0}^{r} \exp\left(2\pi \int_{r_0}^{r} \frac{dr}{\theta(r)}\right) dr = 0,$$

then $f(p)$ is constant. The author also points out connections of these results with related theorems of Noshiro [loc. cit.], Kuramochi [Osaka Math. J. **5**, 155–201 (1953); MR **15**, 518], Pfluger [Ann. Acad. Sci. Fenn. Ser. A. no. **64** (1949); MR **11**, 342] and Kusunoki [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 61–66 (1953); MR **15**, 519].

L. Sario (Los Angeles, Calif.).

Mori, Akira. A note on unramified abelian covering surfaces of a closed Riemann surface. J. Math. Soc. Japan **6**, 162–176 (1954).

Consider a relatively unramified covering surface $\bar{F}$ of a closed Riemann surface F, defined by $p=f(\bar{p})$, $p \in F$, $\bar{p} \in \bar{F}$, say. The following properties are postulated: (a) The inverse function $f^{-1}(p)$ can be continued along each arc on F. (b) For any closed curve α on F, the curves constituting $f^{-1}(\alpha)$ are all closed or all open. (c) The group $\Gamma(\bar{F})$ of projection-preserving self-mappings of $\bar{F}$ is abelian.

Denote by O_G, O_{AB}, and O_{AD} the classes of Riemann surfaces not admitting Green's functions, nonconstant analytic functions that are bounded, and those that possess a finite Dirichlet integral, respectively. The author establishes the following results: (1) $\bar{F} \in O_G$ if and only if the rank of $\Gamma(\bar{F})$ is ≤ 2. (2) $\bar{F} \in O_{AD}$ for any $\Gamma(\bar{F})$ [cf. Sario, Ann. Acad. Sci. Fenn. Ser. A. I. no. **50** (1948); MR **10**, 365]. (3) If the nondividing cycles C_{2i-1}, C_{2i} ($i=1, \cdots, p$) on $\bar{F}$ form a system of generators of $\Gamma(\bar{F})$, and the relations $\gamma_{2i-1}C_{2i-1}-\gamma_{2i}C_{2i}=0$ hold with nontrivial integral coefficients, then $\bar{F} \in O_{AB}$. The

starting points of the author's reasoning are existence tests of Nevanlinna [ibid. **54**, no. 3 (1940); MR **2**, 85, 276], Pfluger [C. R. Acad. Sci. Paris **230**, 166–168 (1950); MR **11**, 342], Royden [Trans. Amer. Math. Soc. **73**, 40–94 (1952); MR **14**, 167], and Sario [Duke Math. J. **20**, 279–286 (1953); MR **14**, 969]. *L. Sario.*

*Martinelli, E. **Sur l'extension des théorèmes de Cauchy aux fonctions de plusieurs variables complexes.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 109–124. Georges Thone, Liège; Masson & C^ie, Paris, 1953.

Exposé des résultats obtenus par l'auteur depuis 1937, sur l'extension des théorèmes de Cauchy. Les résultats les plus généraux se trouvent dans Ann. Mat. Pura Appl. (4) **34**, 277–347 (1953); MR **15**, 304. Des généralisations récentes du premier théorème, par d'autres auteurs, sont signalées. La relation entre le second théorème et l'intégrale d'A. Weil [Math. Ann. **111**, 178–182 (1935)] n'est pas connue. *P. Dolbeault* (Paris).

Hervé, Michel. Itération des transformations analytiques dans le bicercle-unité. Ann. Sci. Ecole Norm. Sup. (3) **71**, 1–28 (1954).

A brief account of the principal results of this paper was given in a previous note [C. R. Acad. Sci. Paris **237**, 1484–1485 (1953); MR **15**, 520]. Let $x_1=f(x, y)$, $y_1=g(x, y)$ be an analytic transformation t of the unit bicylinder into itself. By iteration of this transformation a normal sequence of analytic mappings is obtained. Let τ be an arbitrary limit mapping of this sequence. If t has no fixed points in the interior of the unit bicylinder, the function f must satisfy one of the following two conditions: i) there exists a point $e^{i\alpha}$ such that $f(x, y)$ always belongs to the closed circular disc bounded by a circle through x and touching the unit circle in $e^{i\alpha}$ or ii) there exists an analytic function $x=\xi(y)$, $|y|<1$ such that the relation $f(x, y)=x$ is equivalent to the relation $x=\xi(y)$. An analogous statement with x and y interchanged holds for g. If f satisfies i) and g satisfies ii), all transformations τ have the form $x_1=e^{i\alpha}$, $y_1=\psi(x, y)$, where α but not ψ is uniquely determined. If both f and g satisfy i) the same result may hold, otherwise the corresponding result with x and y interchanged will hold for all τ. If both f and g satisfy ii), the sequence of iterated functions is convergent, and the limit function τ has the form $x_1=e^{i\alpha}$, $y_1=e^{i\beta}$. *H. Tornehave* (Virum).

Thimm, Walter. Über meromorphe Abbildungen von komplexen Mannigfaltigkeiten. Math. Ann. **128**, 1–48 (1954).

Entre $n+1$ fonctions méromorphes de n variables, à $2n$ périodes, il existe une relation algébrique. Ce résultat est établi à la suite d'une étude très précise des représentations méromorphes d'une variété compacte et il est généralisé sous la forme: des fonctions méromorphes sur une variété R compacte, à structure analytique complexe, analytiquement dépendantes sur R, sont algébriquement dépendantes.

La démonstration reprend l'étude des variétés analytiques complexes pour lesquelles l'auteur utilise sa notion de familles particulières (Schar) dans un domaine développée ailleurs [Math. Ann. **125**, 145–164 (1952); MR **15**, 210]. Une étude précise de la représentation locale d'une variété analytique complexe est faite pour discuter l'existence de plusieurs types de "représentations canoniques" dans des domaines particuliers; cette étude est indépendante des résultats de R. Remmert et K. Stein [Math. Ann. **126**, 263–306 (1953); MR **15**, 615]; elle est transposée sur une variété à structure analytique complexe. Signalons: si $K(P)$ est un germe premier en P, $M(P)$ un ensemble analytique en P, définis dans un voisinage X de P où $K(P)$ et les germes premiers de $M(P)$ possèdent des représentations canoniques, si $K(P)$ a un germe premier en $P' \in X \cap K(P) \cap M(P)$ qui est contenu dans $M(P)$, $K(P)$ est contenu dans $M(P)$. Cette étude assez longue, dont on ne peut rendre compte ici, permet de préciser l'existence des composantes connexes des ensembles $A_k^{-1}(z)$, $z \in Z$, qui n'appartiennent pas aux points d'indétermination d'une représentation méromorhep $z=(z_1, \cdots, z_k)=A_k(u)$, $u=u_1, \cdots, u_n$; l'auteur appelle variétés de fibres, sous certaines restrictions, ces composantes. On aboutit ainsi à la proposition essentielle: si R est une variété compacte à structure analytique complexe, $z=A_k(u)$ une image méromorphe de R, s'il existe un domaine $z \in Z$ avec variété de fibres, si w est uniforme sur les fibres correspondant à un point "reguliér" z' de la transformation A_k, alors w est lié algébriquement à $z_1, \cdots, z_k$; le degré en w de la relation est borné en fonction du nombre de fibres de A_k, donc indépendamment de w; la démonstration de l'énoncé s'apparente à celle donnée dans un travail précédent [ibid. **125**, 264–283 (1953); MR **15**, 210]. Cette proposition conduit aisément à la généralisation du théorème de Weierstrass énoncée au début. *P. Lelong* (Paris).

Thimm, Walter. Meromorphe Abbildungen von Riemannschen Bereichen. Math. Z. **60**, 435–457 (1954).

L'auteur étend à des "domaines de Riemann" les résultats du mémoire analysé ci-dessus. Par domaine élémentaire (R) construit au-dessus de C^n comme espace de base, avec variété de ramification projetée en $V=[V(x_1, \cdots, x_n)=0]$, $V(x)$ holomorphe, l'auteur entend une pseudo-multiplicité $\mathfrak{X}$ au-dessus d'un complexe simplicial X de C^n, avec projection A d'un simplexe de $\mathfrak{X}$ sur un simplexe de X; le relèvement A^{-1} fait correspondre à un simplexe de $X-V$ exactement r simplexes de $\mathfrak{X}$ et à un simplexe de V au plus r simplexes de $\mathfrak{X}$; A est localement topologique hors de la variété de ramification $W=A^{-1}(V)$. Un voisinage élémentaire (R) de $0' \in A^{-1}(0)$ se définit par relèvement d'un voisinage de 0 formé de simplexes ayant 0 pour sommet. Soit un complexe $K \subset X$ formant une pseudo-variété connexe à s dimensions, de frontière sur la frontière de X; si $K \cap V$ n'a que des simplexes de dimension au plus $s-2$, pour $P \in K-V$ et $P' \in A^{-1}(P)$, il existe dans $\mathfrak{X}$ une pseudo-multiplicité unique $K' \in A^{-1}(K)$ dont la frontière est sur la frontière de $\mathfrak{X}$ et qui contient P'; si K', K'' sont deux pseudo-multiplicités connexes dans $\mathfrak{X}$ avec la même projection et si $K' \neq K''$, $K' \cap K''$ est dans W. La question de savoir si de tels domaines ramifiés peuvent être toujours continus dans un domaine d'holomorphie est simplement signalée. On construit des germes premiers d'ensembles analytiques par relèvement des germes E dans X non contenus dans V (comme plus haut cette construction est présentée indépendamment de l'obtention de tels germes comme zéros communs à des fonctions holomorphes sur $\mathfrak{X}$). Les ensembles analytiques M qu'on considérera sur $\mathfrak{X}$ seront astreints à quelques restrictions de situation par rapport à W. Un domaine (R) est défini enfin comme espace topologique ayant comme bases de "voisinages (R) distingués" $U_k=T_k^{-1}(\mathfrak{X}_k)$, images topologiques (par T_k^{-1}) d'un voisinage élémentaire (R) $\mathfrak{X}_k$ avec la condition que si $U_1 \cap U_2=S$ est non vide, $T_1(U_1)=\mathfrak{X}_1$, $T_2(u_2)=\mathfrak{X}_2$, $\theta=T_2T_1^{-1}$ soit une représentation holomorphe. L'étude des "familles particulières" d'ensembles analytiques, des repré-

sentations canoniques, et des représentations méromorphes est alors transposée aux domaines (R) pour aboutir au théorème principal: Si des fonctions méromophes sur un domaine (R) compact sont analytiquement dépendantes, elles sont algébriquement dépendantes. *P. Lelong.*

Nisigaki, Hisami. A theory of quaternion-functions, by classifications. III. Mem. School Sci. Engrg. Waseda Univ., Tokyo no. 18, 108–109 (1954).

For part II see same Mem. no. 16 (1952); MR 14, 633.

Theory of Series

***Fempl, Stanimir. Teorija redova za studente viših pedagoških škola i za nastavnike. [Theory of series for students of higher pedagogical schools and for teachers.]** Naučna Knjiga, Belgrade, 1954. iv+135 pp.

An elementary text on series. Ch. 1. Series with constant terms. Ch. 2. Series with variable terms. Ch. 3. Applications of series. Ch. 4. Elementary functions in the domain of complex numbers.

Bagemihl, F., and Erdös, P. Rearrangements of C_1-summable series. Acta Math. 92, 35–53 (1954).

For any C_1-summable series (with real terms) the set of the C_1-sums of the rearrangements of the series consists either of all real numbers or of all numbers of the form $\alpha+U\beta$ $(U=0, \pm1, \pm2, \cdots)$ (for some fixed real α and β). An example for the second case with $\alpha=0$, $\beta=1$ is given by the series $\sum a_n$ with

$$a_n=2^{2^k-k} \text{ if } n=2^{2^k}, \quad a_n=-2^{2^k}-1 \text{ if } n=2^{2^k}+1,$$
$$a_n=1 \text{ if } n=2^{2^k}+2, \quad a_n=0 \text{ otherwise.}$$

S. Mazur [Soc. Sav. Sci. Lett. Lwow 4, 411–424 (1929)] has obtained similar results. *K. Zeller* (Tübingen).

Meili, Heino Jürg. Über das Eindeutigkeitsproblem in der Theorie der asymptotischen Reihen. Comment. Math. Helv. 29, 93–96 (1955).

This paper contains a short synopsis of the author's dissertation, "Über das Eindeutigkeitsproblem in der Theorie der asymptotischen Reihen" [Druckerei Leeman AG, Zürich, 1954]. It is concerned with generalisations of the results of Carleman [Les fonctions quasi analytiques, Gauthier-Villars, Paris, 1926] and Ostrowski [Acta Math. 53, 181–266 (1929)], using the Ahlfors distortion theorem [Acta Soc. Sci. Fenn. Nova Ser. A. 1, no. 9 (1930)]. It includes an extension of these results to the so-called Warschawski region [Math. Z. 35, 321–456 (1932)]. As pointed out by the author, some of the results have been anticipated by S. Mandelbrojt [Séries adhérent, régularisation des suites, applications, Gauthier-Villars, Paris, 1952; MR 14, 542]. *R. Wilson* (Swansea).

van der Corput, J. G. Asymptotics. I. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 206–217 (1954).

This first part is restricted to "pure asymptotics" (abbr.: p.a.), the investigation of expansions in which it is only known that the remainder term possesses at most a certain suitable order of magnitude. A variable ω is introduced, either real, complex or a point lying on a Riemann surface. The following convention is adopted: each number and each function which occurs in p.a. and which is not called "fixed", may depend on ω, whereas fixed means independent of ω. A is called "asymptotically finite" if there exists a fixed real number p such that $A=O(|\omega|^p)$. A and B are called "asymptotically equal": $A\sim B$, if, for each fixed real number q, $A-B=O(|\omega|^{-q})$.

Let $A_1, A_2, \cdots$ be given numbers. A_k is said to tend, as $k\to\infty$, asymptotically to an asymptotic limit s if there exists a fixed integer $k\geqq 0$ such that it is possible to find for each fixed integer $k\geqq k$ a fixed real number q_k with the two properties: $q_k\to\infty$ as $k\to\infty$; for each fixed integer $k\geqq k$, $A_k=s+O(|\omega|^{-q_k})$. The series $a_0+a_1+\cdots$ is said to "converge asymptotically" if it is possible to find a number s such that $\sum_{n=0}^{k} a_n$ tends for $k\to\infty$ asymptotically to s (asymptotic sum $\sum_{k=0}^{\infty}{}' a_k\sim s$). The fundamental theorem of asymptotic convergence is: $a_0+a_1+\cdots$ converges asymptotically if and only if $a_k\to 0$ asymptotically for $k\to\infty$.

$$\sum_{k=0}^{\infty}{}' a_k\sim s, \quad \sum_{k=0}^{\infty}{}' b_k\sim t \quad \text{imply} \quad \sum_{k=0}^{\infty}{}' (a_k\pm b_k)\sim s\pm t; \quad st\sim\sum_{k=0}^{\infty}{}' c_k,$$

where $c_k=a_0b_k+a_1b_{k-1}+\cdots+a_kb_0$ (if a_k and b_k are asymptotically finite for each fixed integer $k\geqq 0$).

Formal substitution: $A\sim\sum_{n=0}^{\infty}{}' a_ns^n$, where

$$s\sim\sum_{k=1}^{\infty}{}' b_kz^k \quad \text{implies} \quad t\sim\sum_{n=0}^{\infty}{}' c_nz^n;$$

here $c_0=a_0$, c_n $(n\geqq 1)$ is defined by the formal identity

$$\sum_{n=1}^{\infty} c_nz^n=\sum_{k=1}^{\infty} a_k\Big(\sum_{k=1}^{\infty} b_kz^k\Big)^k,$$

if there exists a fixed real p and a fixed $\delta>0$ such that for each fixed positive integer k, $a_k=O(|\omega|^p)$ and $b_kz^k=O(|\omega|^{-\delta})$. If the terms a_k $(k=0, 1, \cdots)$ and the number s depend on one or more parameters, and $\sum_{n=0}^{k-1} a_n$ tends for $k\to\infty$, uniformly in these parameters, asymptotically to s, then the series $a_0+a_1+\cdots$ is said to converge "asymptotically uniform" in these parameters. The paper terminates with some theorems about analytic functions with prescribed asymptotic expansion, and asymptotic expansion of a root s of the equation $f(s)=t$ ($f(s)$ continuous in the interval $|s|\leqq\beta$, such that $f(s)=$ as $\{1+O(|\omega|^{-\delta}\}$ uniformly in s; a independent of s, $|a|\geqq\gamma\beta^{-1}$, $\gamma>0$). In this paper no proofs are given [for these proofs cf. Asymptotic expansions, I, Dept. of Math., Univ. of California, 1954; MR 16, 352]. *S. C. van Veen* (Delft).

Blambert, Maurice. Quelques théorèmes Faberiens relatifs au problème Hadamard-Mandelbrojt. Rend. Circ. Mat. Palermo (2) 3, 214–243 (1954).

In this paper the author continues his previous investigations on the composition of singularities of Dirichlet series [Acta Math. 89, 217–242 (1953); MR 15, 206]. Conditions are given under which the point $\gamma=\alpha+\beta$, where α is an "accessible" singularity of $f(s)=\sum a_ne^{-\lambda_ns}$ and β is an essential singularity of $\varphi(s)=\sum b_ne^{-\mu_ns}$, is a singularity of the Mandelbrojt-Hadamard composed series

$$H_M[f(s), \varphi(s)|k]=\sum a_n^{(k)}b_ne^{-\mu_ns},$$

hence generalizing well known results of Faber and Pólya in the case of the Hadamard composition of Taylor series. Results on the singularities of the Cramer transform of $\varphi(s)$, $\sum b_n\theta(\mu_n)e^{-\mu_ns}$, are also announced. Finally, a case is discussed where there is an infinity of couples α_i, β_i such that $\alpha_i+\beta_i=\gamma$, and where it can be asserted that γ is a singularity of $H_M[f(s), \varphi(s)|k]$ for k sufficiently large.

S. Agmon (Jerusalem).

Fourier Series and Generalizations, Integral Transforms

Devinatz, A. Integral representations of positive definite functions. II. Trans. Amer. Math. Soc. **77**, 455–480 (1954).

The paper extends the results of the first part [same Trans. **74**, 56–77 (1953); MR **14**, 659], using the theory of reproducing kernels to unify the theories of positive definite and of completely monotonic functions, and to extend these theories to functions of several variables. In the present paper, functions defined over a convex set, not necessarily the entire space, are dealt with.

Let Z_{m+n} denote the set of points z in $(m+n)$-dimensional complex space of form (x, iy), $x \in E_m$, $y \in E_n$; $\mathrm{Re}(z)=x$, $\mathrm{Im}(z)=y$. The first theorem extends a theorem of the earlier paper. Let P be a set in Z_{m+n} for which $\mathrm{Re}(z)$ runs over an interval in E_m containing the origin. Let $a=(d, -c)$, $b=(e, c)$, where $d^k \leqq e^k$, $c_k \geqq 0$, be in E_{m+n}. Then conditions are given that a function $f(z)$ defined on P have a representation there of the form $\int_a^b e^{z\cdot t} d\alpha(t)$, where α is a non-negative bounded measure on E_{m+n}.

Let Q be an open convex set in the complex plane which contains the origin and is symmetrical with respect to the real axis. Necessary and sufficient conditions are given for a function $f(z)$ to have on Q a representation $\int_{-c}^{c}\int_a^b e^{zt_1+iwt_2} d\alpha(t_1, t_2)$, where α is a bounded non-negative measure, and $-\infty \leqq a < b \leqq \infty$, $0 < c \leqq \infty$. The classical theorems on positive definite and completely monotonic functions are included as special cases, as is also a corrected version of a theorem of Livshitz [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **44**, 3–7 (1944); MR **6**, 131] which is shown by an example to be incorrect in its original form.

Let Q' be an open convex set in Z_{m+n} containing the origin, such that $\mathrm{Re}(Z_{m+n})$ fills an interval in E_m and such that if $z \in Q'$, so are the vectors formed from z by replacing z^k by $-z^k$ for each $k \geqq m+1$. Let $a=(d, -c)$, $b=(e, c)$, where c, d, and e are as above but c must be finite. Then necessary and sufficient conditions are given for a function $f(z)$ to have on Q' a representation $\int_a^b e^{z\cdot t} d\alpha(t)$.

Lastly, a double moment problem is solved: necessary and sufficient conditions are given that a double sequence $\{\mu_{m,n}\}$, $m=0, 1, \cdots$, $n=0, \pm 1, \pm 2, \cdots$, have a representation $\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} u^m e^{imv} d\alpha(u, v)$ with non-negative bounded measure α. *J. L. B. Cooper* (Cardiff).

Boas, R. P., Jr. Order of magnitude of Fourier transforms. Michigan Math. J. **2** (1953–54), 141–142 (1955).

Let $f(x)$ and $g(x)$ be a pair of Fourier transforms. It is proved that, if both are $O(\exp[\tau|x|^\rho - \frac{1}{2}x^2])$ as $|x| \to \infty$, where $0 < \rho < 2$, then both are of the form $\Phi(x)\exp[-\frac{1}{2}x^2]$ where $\Phi(x)$ is an integral function, at most of order ρ and finite type. This generalizes an earlier result of Hardy [see Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1937, p. 175]. *W. W. Rogosinski.*

Blackman, Jerome. The inversion of the generalized Fourier transform by Abelian summability. Trans. Amer. Math. Soc. **78**, 19–29 (1955).

The author presents an inversion formula and a theorem of the Hausdorff-Young type for generalized Fourier transforms. Let L_p^k be the class of functions ϕ such that $\phi(t)(1+|t|^k)^{-1}$ is in L^p and let $\|\phi\|_{k,p}$ denote the L^p-norm of $\phi(t)(1+|t|^k)^{-1}$. Then if $E(h, x)$ is the kth Fourier transform of ϕ in the sense of Bochner [Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932, ch. VI], the function

$$\phi_\epsilon(t) = (-1)^k(2\pi)^{-1}\int_{-\infty}^{+\infty} (-\epsilon \operatorname{sg} x - it)^k E(h, x) \times \exp x(-\epsilon \operatorname{sg} x - it)\, dx$$

converges in norm $\| \ \|_{k,p}$ to ϕ as $\epsilon \to 0$ provided that $h=k$ if $p=1$, or that $h=k+1$ if $1<p<\infty$ or that $h=k+2$ if $p=\infty$. If $1<p\leqq 2$ and $k \geqq 0$, then

$$\|E(k, x)\|_{k,q} \leqq (2\pi)^{-1/q-1/2}\|\phi(t)\|_{k,p},$$

where $p^{-1}+q^{-1}=1$, which generalizes the theorem of Hausdorff-Young. *A. P. Calderón* (Columbus, Ohio).

Doetsch, Gustav. Caratterizzazione della trasformazione di Laplace mediante la relativa regola di derivazione negli spazi L^p e U. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 444–449 (1954).

Der Verfasser beweist den folgenden Satz: Ein distributives Funktional $T(F)$ das von einem komplexen Parameter s abhängt und welches für ein solches s mit $\Re(s)>0$ stetig ist in einem Raume L^p oder U und für die Folge $\phi_n(t)=e^{-t/2}t^n$ $(n=0, 1, 2, \cdots)$ der Differentiationsregel

$$T(F')=sT(F)-F(+0)$$

genügt, muss eine L-Transformation in L^p bzw. in U sein. *W. Saxer* (Zürich).

***Pleijel, Arne. Beitrag zur Theorie der Laplace-Transformationen.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 217–221 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The following theorem is proved: Let $f(s)=\int_0^\infty e^{-sx}F(x)dx$ converge for some point $s=s_0$ $(\mathrm{Re}\{s_0\}>0)$. A necessary and sufficient condition for $f(s)$ to be also analytic in the half-plane $\mathrm{Re}\{s\}>0$, and satisfy there an inequality of the form: $|f(s)| \leqq c(1+|t|)^k$ $(s=b+it)$, is that there exist a constant K such that:

$$\left|\int_0^y e^{-s\tau}F(x)(y-x)^{k+1}dx\right| \leqq Ky^{k+1}(1+|\tau|)^k$$

for all $y>0$ and real τ. *S. Agmon* (Jerusalem).

Rooney, P. G. On an inversion formula for the Laplace transformation. Canad. J. Math. **7**, 101–115 (1955).

Let $f(s)=\int_0^\infty e^{-st}\phi(t)dt$. The author considers the following operator given by Erdélyi:

$${}^\nu L_{k,l}[f(s)] = \frac{k^{3/2}e^{2k}}{t\pi^{1/2}}\int_0^\infty x^{\nu/2}J_\nu(2kx^{1/2})f\left(\frac{k(x+1)}{t}\right)dx$$

and shows that under suitable conditions it is a "real" inversion operator: $\lim {}^\nu L_{k,l}[f(s)]=\phi(t)$ $(k\to\infty)$. The special case $\nu=-\frac{1}{2}$ was already considered by the author [same J. **4**, 436–444 (1952); MR **14**, 554]. The operator is also used to derive necessary and sufficient conditions for a function to be equal almost everywhere to a Laplace transform of a function of the form $t^\alpha\phi(t)$, $\phi \in L_p(0, \infty)$, $1<p\leqq\infty$, $\alpha > -(p-1)/p$. *S. Agmon* (Jerusalem).

Hull, T. E., and Froese, C. Asymptotic behaviour of the inverse of a Laplace transform. Canad. J. Math. **7**, 116–125 (1955).

In theorem 1 the authors give sufficient conditions for $f(s)$ to insure that the inverse Laplace transform

$$F(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}f(s)ds$$

exists (as a principal value) and can be evaluated by deforming the path of integration into the left half-plane. In theorem 2 they prove that in the case that $F(t)=0$ for $0<t<1$ and $F(t)=\Phi(t)$ for $t>1$, where $\Phi(t)$ is one of a number of standard comparison functions which they list, $f(s)-\phi(s)$ is an entire function. They explain the use of these theorems for obtaining asymptotic expansions of inverse Laplace transforms, and illustrate their process by determining the asymptotic behavior of $F(t)$ as $t\to\infty$ in the cases when $f(s)$ is

$$\frac{\log s}{1+s^2},\quad \frac{\exp(-1/s)}{s(1+s)},\quad \frac{1}{(1-s)\log s}.$$

The paper concludes with an asymptotic expansion of

$$\int_a^{a+i\infty}\frac{t^{u-1}}{\Gamma(u)}du-e^t$$

as $t\to\infty$. *A. Erdélyi* (Pasadena, Calif.).

Bose, B. N. Certain theorems on self-reciprocal functions. Bull. Calcutta Math. Soc. **46**, 109–127 (1954).

The author presents a number of theorems of which the first one is: Let

$$p^{m+1}\int_0^\infty f(t)(t^2+p^2)^{-m-1}dt \quad (m>0)$$

be R^m; then $x^{-m-1}f(x)$ is also R_m, provided the latter function is continuous and absolutely integrable in $(0,\infty)$.

A. Erdélyi (Pasadena, Calif.).

***Obreschkoff, Nikola. Über Integraldarstellungen reeller Funktionen auf der positiven Halbgeraden.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 197–202. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

This is a brief report on some of the author's work on the Meijer integral transform. [Cf. Bülgar. Akad. Nauk. Izv. Mat. Inst. **1**, 83–110 (1953); MR **15**, 119, and references given in that review.] *R. P. Boas, Jr.*

Guinand, André Paul. Généralisation d'une identité de M. Halphen. C. R. Acad. Sci. Paris **240**, 582–583 (1955).

The generalized identity is

$$\int_0^\infty f(x)\sin(ax+\theta)x^{s-1}\,dx = \frac{\Gamma(s)}{\sqrt{2\pi}}\int_0^\infty\left[g(x+a)\sin\left(\frac{s\pi}{2}-\theta\right)+g(x-a)\sin\left(\frac{s\pi}{2}+\theta\right)\right]x^{-s}\,ds,$$

where $g(x)$ is the Fourier cosine transform of $f(x)$, $0<s<1$, $f(x)$ is integrable over $(0,\infty)$, and $x^{-s}f(x)$ is integrable at 0.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Tomić, Boško S. Sur les intégrales se rattachant aux polynomes $A_r^n(\alpha)$ et aux fonctions rationnelles $\varphi_n(a,x)$. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 97–108 (1954). (Serbo-Croatian summary)

Put

$$A_r^n(\alpha)=\sum_{i=0}^{r}(-1)^i\binom{n+1}{i}(\alpha+r-i)^n,\quad A_r^n=A_r^n(0).$$

The numbers A_r^n were introduced by Euler [Institutiones calculi differentialis, vol. 2, 2nd ed., Galeatii, Ticini, 1787] and have been discussed by many writers, in particular Worpitzky [J. Reine Angew. Math. **94**, 203–232 (1883)] and Frobenius [S.-B. Preuss. Akad. Wiss. **1910**, 809–847]. The present paper contains many properties of $A_r^n(\alpha)$ and A_r^n including relations between these quantities and the numbers of Bernoulli, Euler and Stirling. The author also discusses certain definite integrals related to $A_r^n(\alpha)$; in particular, he proves

$$\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^{2n+1}\cos(n-2r+1)t\,dt=\frac{\pi}{n!}A_r^n.$$

L. Carlitz (Durham, N. C.).

Davis, Philip, and Pollak, Henry. Complex biorthogonality for certain sets of polynomials. Duke Math. J. **21**, 653–667 (1954).

Synthèse couvrant différents ensembles de fractions rationnelles et de polynomes utilisés dans la représentation des fonctions holomorphes dans un domaine et y appartenant à la classe L^2; ces ensembles étant construits à partir d'une famille complète de fonctionnelles linéaires. Application aux polynomes de Tchebycheff (dans les domaines limités par une ellipse), aux fractions rationnelles de Takenaka-Walsh (dans le cercle unité), à des polynomes du type de Faber (dans un domaine à frontière analytique). *J. Favard.*

Koschmieder, Lothar. Orthogonal polynomials of certain simple domains in the plane and in space. Tecnica. Rev. Fac. Ci. Ex. Tec. Univ. Nac. Tucuman **1**, 173–181 (1951). (Spanish. English summary)

Multiplicative representations of orthogonal polynomials belonging to the circular disk and the solid sphere are summarized. Here these formulas are generalized and analogous results for the rectangular isosceles triangle and the corresponding tetrahedron are given. *E. Frank.*

Motzkin, T. S., and Walsh, J. L. Least pth power polynomials on a real finite point set. Trans. Amer. Math. Soc. **78**, 67–81 (1955).

The authors consider polynomials

$$T_{n+1}(z)=z^{n+1}+\sum_1^{n+1}b_\nu z^{n+1-\nu}$$

which minimise a deviation from zero of the form

$$\delta(|T_{n+1}(z_1)|,\cdots,|T_{n+1}(z_m)|)$$

and the z_k form a finite real point set E; in most of the cases the T_{n+1} are not unique. For a general class of δ, between two zeros of a T_{n+1} lies a point of E; numerous other properties are proved. In the case $\delta=\sum\mu_k|T_{n+1}(z_k)|$, $\mu_k>0$, the totality of T_{n+1} form a convex set generated by basic polynomials having their zeros at points of E; the set of T_{n-1} can be found explicitly, even if E is not real. If $\delta=\sum\mu_k|T_{n+1}(z_k)|^p$, $p>1$, a separation property holds for the zeros of T_n, T_{n-1}. Other results connect best approximations with interpolation, covering also cases of approximation by families of functions more general than polynomials.

F. V. Atkinson (Oxford).

Wenzl, F. Über Gleichungssysteme der Tschebyscheffschen Approximation. Z. Angew. Math. Mech. **34**, 385–391 (1954). (English, French and Russian summaries)

On sait que le problème d'approximation de Tchebycheff, pour certaines familles de fonctions $f(x;a_1,\cdots,a_n)$ définies

sur $[A, B]$, se ramène à la résolution du système

$$\text{(S)}\quad \begin{aligned} &A=\xi_1<\xi_2<\cdots<\xi_{n+1}=B,\\ &f(\xi_k; A_1, \cdots, A_n)=(-1)^k L \quad \text{pour} \quad k=1, \cdots, n+1,\\ &|f(x; A_1, \cdots, A_n)|\leq L \quad \text{pour} \quad A\leq x\leq B. \end{aligned}$$

Si f est une fraction rationnelle en x, l'auteur remarque que (S) équivaut à un système (S′) où n'interviennent plus les ξ_k, mais les fonctions symétriques de $\xi_2, \xi_4, \cdots$, et de $\xi_3, \xi_5, \cdots$. Il montre, sur deux exemples, que (S′) peut se prêter, beaucoup mieux que (S), au calcul numérique des A_k.

J. P. Kahane (Montpellier).

Merli, Luigi. Una formula di approssimazione asintotica per i polinomi di Tchebychef-Hermite e valutazione numerica del resto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 611–614 (1954).

It is known that for real x

$$H_n(x)=f(x, n)\sum_{\nu=0}^{p-1}u_\nu(x)N^{-\nu}+g(x, n)\sum_{\nu=0}^{p-1}v_\nu(x)N^{-(\nu+\frac{1}{2})}+R_p(n, x)N^{-p},$$

where $N=2n+1$, $f(x, n)$ and $g(x, n)$ are known functions, and $u_\nu(x)$ and $v_\nu(x)$ are polynomials [G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., vol. 23, New York, 1939, p. 194; MR 1, 14]. The author determines $u_1(x)$ and $v_1(x)$ and gives an explicit bound for $R_2(n, x)$ when $x>0$.

N. D. Kazarinoff (Lafayette, Ind.).

Orts, J. M.ª On some transformations of Legendre series. Rev. Mat. Hisp.-Amer. (4) 14, 204–209 (1954). (Spanish)

In a previous note [same Rev. (4) 12, 201–205 (1952); MR 14, 466] the author gave a formula for the sum of a series of Legendre polynomials

$$(1)\qquad \sum_{n=0}^{\infty}a_nP_n(x)=\frac{1}{\pi i}\int_{x-i\sqrt{(1-x^2)}}^{x+i\sqrt{(1-x^2)}}\frac{\sum a_nz^n dz}{\sqrt{(1-2xz+z^2)}},$$

where the function $f(z)$ defined by the series $\sum a_nz^n$ is known. Formula (1) is valid for those values of x corresponding to the points interior to the ellipse of convergence of the series $\sum a_nP_n(x)$. Because of difficulties in the calculation of the integral (1), the author gives in the present paper the following transformations of the series $\sum a_nP_n(x)$. I. From the formula

$$\sum_{n=0}^{\infty}a_nP_n(x)=\frac{1}{\pi}\int_0^{\pi}[\sum a_n(x+(x^2-1)^{1/2}\cos\phi)^n]d\phi$$

which results from (1) by the change of variable

$$z=x+(x^2-1)^{1/2}\cos\phi,$$

one obtains

$$\sum_{n=0}^{\infty}a_nP_n(x)=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}f^{(2n)}(x)(1-x^2)^n.$$

II. From the formula

$$P_n\left(\frac{1+x}{1-x}\right)=\frac{1}{(1-x)^n}\sum_{k=0}^{n}\binom{n}{k}^2x^k,$$

one obtains from (1)

$$\sum_{n=0}^{\infty}a_nP_n(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}a_n\left[\sum_{k=0}^{n}\binom{n}{k}^2(x-1)^k(x+1)^{n-k}\right].$$

E. Frank (Chicago, Ill.).

Special Functions

***Erdélyi, Arthur, Magnus, Wilhelm, Oberhettinger, Fritz, and Tricomi, Francesco G. Higher transcendental functions. Vol. III. Based, in part, on notes left by Harry Bateman.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. xvii+292 pp. $6.50.

This is the final volume on the higher transcendental functions prepared at the California Institute of Technology by the staff of the Bateman Manuscript Project; the first two volumes appeared in 1953 [MR 15, 419]. Actually, this volume was prepared by Erdélyi after the staff of the project left Pasadena, but the book owes a great deal to Magnus, who continued to work on it.

The topics discussed are automorphic functions, Lamé functions, Mathieu functions and spheroidal and ellipsoidal wave functions, functions of number theory, miscellaneous functions, generating functions. All is done in the thorough well-documented style of the earlier volumes.

E. T. Copson (St. Andrews).

***Šerbatov, V. G. Giperboličeskie funkcii. [Hyperbolic functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 56 pp. .80 rubles.

No. 16 in the series, Popular lectures on mathematics.

Cazenave, René. Représentation géométrique de la transformation de Gudermann. Ann. Télécommun. 9, 330–333 (1954).

Ascoli, Guido. A proposito di alcuni recenti risultati asintotici sugli estremi di funzioni speciali. Univ. e Politecnico Torino. Rend. Sem. Mat. 13, 271–283 (1954).

In the differential equation $\sum_{s=0}^{n}x^{n-s-1}p_s(x, t)y^{(n-s)}=0$ let x be a complex variable, t a real or complex parameter varying over a bounded closed set I containing $t=0$ as a limit point; and let $p_0=1$, p_s continuous for $t\in I$ and $|x|<R$, and for each $t\in I$ regular for $|x|<R$. Clearly, $x=0$ is a regular singularity, and one of the roots of the indicial equation at $x=0$ is zero. Assume that no root of the indicial equation at $x=0$ is a positive integer when $t\in I$. Then for each $t\in I$, the differential equation has a unique solution $Y(x, t)$ which is regular for $|x|<R$, and such that $Y(0, t)=1$; and this solution is a continuous function of x and t. The author proves that the zeros, turning points, and extremal values of $Y(x, t)$ qua function of x vary continuously with t. He then proceeds to show that many results on the asymptotic behavior of the zeros, turning points, and extremal values of the classical orthogonal polynomials and of other special functions can be derived from his theorem.

A. Erdélyi (Pasadena, Calif.).

***Meixner, Josef, und Schäfke, Friedrich Wilhelm. Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXXI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xii+414 pp. DM 49.00; clothbound, DM 52.60.

Die Mathieuschen Funktionen und die Sphäroidfunktionen treten u.a. bei der Separation der Schwingungsgleichung $\Delta u+k^2u=0$ in elliptischen Zylinderkoordinaten bzw. in rotationselliptischen Koordinaten auf und stellen Verallgemeinerungen der trigonometrischen Funktionen, Kugelfunktionen und konfluenten hypergeometrischen Funk-

tionen dar. Das vorliegende Werk gliedert sich in vier Hauptteile: 1) Mathieusche Grundlagen; 2) Mathieusche Funktionen (Definitionen, Haupteigenschaften, Entwicklungssätze, Additionstheoreme, Reihenentwicklungen, Integralrelationen); 3) Sphäroidfunktionen; 4) Anwendungen der Mathieuschen und Sphäroidfunktionen.

In 2) wird die Natur der Lösungen $Y(z;\lambda;h^2)$ der Mathieuschen Differentialgleichung

$$(1) \qquad Y''(z)+(\lambda-2h^2\cos 2z)\,Y(z)=0$$

untersucht. Diese Lösungen sind ganze, und im allgemeinen transzendente Funktionen von z. Fundamentalsystem Y_I, Y_{II} mit $Y_I(0)=1$, $Y_I'(0)=0$; $Y_{II}(0)=0$, $Y_{II}'(0)=1$. Floquetsche Lösungen $Y(z)=me_\nu(z)$ mit $Y(z+\pi)=e^{i\pi\nu}Y(z)\neq 0$.

Der charakteristische Exponent ν genügt:

$$\cos\pi\nu=Y_I(\pi;\lambda;h^2).$$

(2) $me_{\pm\nu}(z)=ce_\nu z\pm i\,se_\nu z$. Numerische Berechnung der Eigenwerte λ durch Kettenbruchgleichungen: z.B.

$$\lambda=4-\cfrac{h^4}{4^2-\lambda}-\cfrac{h^4}{6^2-\lambda}-\cdots.$$

Die Lösungen $\lambda=\lambda_\nu(h^2)$ von (2) werden als charakteristische Kurven in der Stabilitätskarte eingezeichnet. Asymptotische Entwicklung: Für $m=0,1,2,\cdots$ gilt bei $h\to+\infty$ im Intervall $(0\leqq z\leqq\pi)$ gleichmässig:

$$ce_m(z;h^2)=\frac{se_{m+1}(z;h^2)}{\sin z}=\left(\frac{\pi h}{2}\right)^{1/4}(m!)^{-1/2}(-1)^m \times\left\{e^{z^2/4}\frac{d^m}{dx^m}e^{-x^2/2}\right\}_{x=2h^{1/2}\cos z}+O(h^{-3/4}).$$

Für reelles h^2 besitzen $ce_m(z;h^2)$, $se_{m+1}(z;h^2)$ $(m=0,1,2,\cdots)$ im Intervall $0<z<\pi$ je genau m Nullstellen, die für $h^2\to+\infty$ asymptotisch den Nullstellen des Hermiteschen Polynoms $He_m(h^{1/2}(\pi-2z))$ nähern. Für $h^2>0$ sind mit z_0 auch $k\pi\pm z_0$, $k\pi\pm\bar z_0$ Nullstellen. Es gibt weiter für $h^2>0$ auch komplexe Nullstellen, die im Streifen $-\frac{1}{2}\pi<\mathrm{Re}\,z\leqq+\frac{1}{2}\pi$ rein imaginär sind. In 3) werden gleicherweise die Sphäroidfunktionen als Lösungen der Gleichung

$$[(1-z^2)Y'(z)]'+\left[\lambda+\gamma^2(1-z^2)-\frac{\mu^2}{1-z^2}\right]Y(z)=0$$

betrachtet. Fundamentallösungen $Y_I(z)$, $Y_{II}(z)$ mit

$$\lim_{z\to 0}(1-z)^{-\mu/2}Y_I(z)=1,\quad \lim_{z\to 0}(1-z)^{\mu/2}Y_{II}(z)=(2\mu)^{-1}.$$

In 4) werden viele Anwendungen gegeben, u.a. mechanische und elektrische Schwingungen mit periodisch veränderlichen Parametern, z.B. das Pendel mit oszillierendem Aufhängepunkt (mögliche Stabilität in der Nähe der labilen Gleichgewichtslage $\alpha=\pi$; Rolle in der Equilibristik), das stark fokussierende Synchrotron, Schwingungen einer elliptischen Membran, Abstrahlungs- und Beugungsprobleme, wellenmechanische Probleme. Das Werk schliesst mit einem ausgedehnten Literaturverzeichnis (mehr als 400 Zitate, meist seit 1920). Im Vorwort geben die Verfasser eine genaue Aufzählung ihrer persönliche Beiträge zu dieser ausgedehnten Arbeit. Diese Beiträge sind vielfach mit einander verknüpft und können daher oft kaum von einander getrennt werden. Wesentliche Anregungen verdankt der erste der beiden Verf. der Dissertation von Bouwkamp [Univ. of Groningen, 1941; MR 8, 179], nebst vielen späteren ergiebigen Diskussionen mit Herrn Bouwkamp.

Die Verfasser möchten die folgenden Druckfehler berichtigen: S. 80, Z. 1*: lese $(f,y_n^*(\bar\mu))y_n(\mu)$ statt $(f,y_n(\bar\mu))y_n^*$; S. 292, Z. 2*: lese $\mu(3)$ statt $(\mu\ 3)$; S. 309, Z. 12*: lese $|\xi-\alpha|>|\alpha\pm 1|$ statt $|\xi-\alpha>|\alpha|\pm 1|$; S. Z. 5*: lese 2.25 statt 2.55; S. 331, Gl. (19): lese $m\bar\jmath$ statt my; S. 357, Z. 2 nach Gl. (5): lese $\Delta u+k^2u=0$ statt $\Delta E+k^2E=0$; S. 260, Gl. (2): lese $z^{\nu-n}$ statt $z^{\nu+n}$; Gl. (5): lese $\binom{n}{r}$ statt $\binom{n}{r}$; S. 204, Gl. (43), Z. 3: lese $\mathrm{Ms}_{2n+1}^{(1)\prime}$ statt $\mathrm{Ms}_{2n+1}^{(1)}$; Gl. (44), Z. 3: lese $\mathrm{Ms}_{2n+1}^{(2)}$ statt $\mathrm{Ms}_{2n+1}^{(2)\prime}$.

S. C. van Veen (Delft).

Meixner, Josef, und Schäfke, Friedrich Wilhelm. Eigenwertkarten der Sphäroiddifferentialgleichung. Arch. Math. **5**, 492–505 (1954).

In the notation of the authors' recent book reviewed above, let $\lambda_\nu^\mu(\gamma^2)$ denote the characteristic values of the spheroid differential equation

$$[(1-z^2)y'(z)]'+\left[\frac{-\mu^2}{1-z^2}+\lambda+\gamma^2(1-z^2)\right]y(z)=0.$$

For any values of the parameters λ, γ^2, μ^2 there exists a spheroid function $y(z)$ with the property that

$$y(ze^{\pi i})=e^{\pi i\nu}y(z),\quad |z|>1,$$

where ν is the characteristic exponent. It is known, for example, that $\sin\pi\nu$ is an entire function of λ, γ^2, μ^2. The paper under review supplements certain sections of the book cited above. In 8 figures, curves are presented of $\lambda_\nu^\mu(\gamma^2)$ for constant ν and fixed μ in the (λ,γ^2)-plane, that is, for $\mu=0$, 1/4, 1/2, 3/4, 0.95, 1, 1.05, 5/4 and in particular for $\nu\equiv\pm\mu \pmod 1$. *C. J. Bouwkamp* (Eindhoven).

Bloh, Ê. L. On an expansion of Bessel functions in a series of Legendre functions. Prikl. Mat. Meh. **18**, 745–748 (1954). (Russian)

A new generating function relation,

$$e^{it}I_m[t(z^2-1)^{1/2}]=\sum_{n=0}^{\infty}\frac{t^{m+n}P_{m+n}^m(z)}{(2m+n)!},$$

for the associated Legendre functions is given. This relation is obtained by termwise integration of the series for $\exp\{t[z+(z^2-1)^{1/2}\cos\varphi]\}$ after multiplication by $\cos m\varphi$. Numerous definite integrals are evaluated through the use of various forms of the generating function relationship. The case $m=0$ was discussed by Rainville [Bull. Amer. Math. Soc. **51**, 268–271 (1945); MR 6, 211].

N. D. Kasarinoff (Lafayette, Ind.).

Nagase, Masahumi. On the zeros of certain transcendental functions related to Hankel functions. I, II. J. Phys. Soc. Japan **9**, 826–841, 842–853 (1954).

The author studies the zeros of functions of ν which are compounded of Bessel functions of order ν and fixed positive argument(s), and of certain other comparatively simple functions. In the first paper the function investigated is a linear combination, with coefficients depending on a and ν, of $H_\nu^{(2)}(a)$, $H_\nu^{(2)\prime}(a)$, where $a>0$ is given. As an example, the zeros of $H_\nu^{(2)\prime}(a)+kH_\nu^{(2)\prime}(a)$ (arising in the theory of diffraction of waves by a sphere) are studied. In the second paper the function is a bilinear combination, with coefficients depending on ν, a, m, of $H_\nu^{(2)}(a)$, $H_\nu^{(2)}(ma)$, $H_\nu^{(2)\prime}(a)$, $H_\nu^{(2)\prime}(ma)$, where $a>0$, $m>0$ are fixed. As an example, the zeros of a function of this form arising in the theory of diffraction of elastic waves by a spherical cavity in a solid body are investigated. *A. Erdélyi* (Pasadena, Calif.).

Ragab, F. M. Linear relations between E-functions and Bessel functions. Acta Math. **92**, 1–11 (1954).

In this paper the author sums six finite series of E-functions in terms of E functions, and gives special cases of his results. *A. Erdélyi* (Pasadena, Calif.).

Chak, A. M. A generalisation of Bessel-Maitland function. Ann. Soc. Sci. Bruxelles. Sér. I. **68**, 145–156 (1954).

The author studies

$$\sum_{n=0}^{\infty}\left[n!\prod_{k=1}^{p}\Gamma(a_k+nb_k+1)\right]^{-1}(-1)^n x^n,$$

obtaining operational forms, series, recurrence relations, etc., and for the case of (non-negative) integer b_k also a differential equation. *A. Erdélyi* (Pasadena, Calif.).

Bose, N. N. On integral representations of MacRobert's E-function. Bull. Calcutta Math. Soc. **46**, 97–101 (1954).

This paper contains a number of integrals involving $E(\alpha, \beta::z)$. [Most, if not all, of the results appear to be paraphrases of known integrals involving confluent hypergeometric functions.] *A. Erdélyi* (Pasadena, Calif.).

Parodi, Maurice. Sur une conséquence de l'emploi de l'analyse symbolique à la résolution d'une équation fonctionnelle. C. R. Acad. Sci. Paris **240**, 144–145 (1955).

The author shows by operational methods that

$$\phi(s)=\frac{\pi d}{\alpha}\operatorname{cosec}\left(\frac{\pi s}{\alpha}\right)\left[\mathbf{J}_{s/\alpha}\left(\frac{2k}{\alpha}\right)-J_{s/\alpha}\left(\frac{2k}{\alpha}\right)\right]$$

satisfies the functional equation

$$s\phi(s)+k[\phi(s-\alpha)+\phi(s+\alpha)]=d,$$

and points out that this result is also a simple consequence of the recurrence relations satisfied by the Bessel function $J_\nu(z)$ and Anger's function $\mathbf{J}_\nu(z)$. *A. Erdélyi.*

Brachman, Malcolm K. Note on an integral of Ramanujan. J. Math. Phys. **33**, 374–375 (1955).

The author reports briefly on the evaluation of the integral

$$\int_{-\infty}^{\infty}\operatorname{sech}(\pi x)\cos(ax)\exp(-bx^2)dx.$$

[See also Erdélyi et al., Tables of integral transforms, vol. I, McGraw-Hill, New York, 1954, p. 35 (43); MR **15**, 868.] *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Ishikawa, Osamu. On the mass distribution generated by a function of P. L. class. Proc. Japan Acad. **30**, 532–537 (1954).

The author proves the following theorem. If $p(x, y)$, $q(x, y)$ and $p(x, y)\,q(x, y)$ are subharmonic in a domain G, and if $\mu_p(e)$ and $\mu_q(e)$ are the mass distributions associated with $p(x, y)$ and $q(x, y)$, respectively, then

$$\lim_{r\to 0}\frac{4}{r^2}[L(p;x,y;r)L(q;x,y;r)-A(pq;x,y;r)]$$

$$=\tfrac{1}{2}pD\mu_q+\tfrac{1}{2}qD\mu_p-\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\frac{\partial q}{\partial y}$$

holds almost everywhere in G. Here L and A denote peripheral and areal circular means, respectively, and $D\mu$ is the symmetric derivative of $\mu(e)$. The author's result implies a generalization of an earlier result due to the reviewer [Bull. Amer. Math. Soc. **53**, 89–95 (1947); MR **8**, 461] concerning functions of class P.L. introduced by Beckenbach and Radó [Trans. Amer. Math. Soc. **35**, 648–664 (1933)]. *M. Reade* (Paris).

Walsh, J. L. An interpolation problem for harmonic functions. Amer. J. Math. **76**, 259–272 (1954).

Designate by H the class of harmonic functions

$$u(z)=\tfrac{1}{2}a_0+\sum_{k=1}^{\infty}r^k(a_k\cos k\theta+b_k\sin k\theta),\quad z=re^{i\theta},$$

with

$$\sum_{k=1}^{\infty}(a_k^2+b_k^2)<\infty.$$

The principal problem of interpolation for the class H is that of solving the infinite system $u(\alpha_k)=A_k$ $(k=0, 1, \cdots)$, where α_k is a set of points interior to $|z|=1$. This problem has been discussed exhaustively for analytic functions with a variety of regions and norms [cf. Walsh, Interpolation and approximation . . . , Amer. Math. Soc. Colloq. Publ., v. 20, New York, 1935; also Bergman, Memor. Sci. Math. no. **106** (1947); MR **11**, 344; Lokki, Ann. Acad. Scient. Fenn. Ser. A. I. Math.-Phys. no. **39** (1947); MR **9**, 277; Walsh and Davis, J. Analyse Math. **2**, 1–28 (1952); MR **16**, 580].

The object of the present paper is to show the application of extremal (orthonormal) techniques to the harmonic case and to indicate what departures of method from the parallel analytic case are required. The principal feature of the method is the use of a sequence of orthogonal functions of class H, $\psi_n(z)$, which satisfy

$$\psi_n(\alpha_0)=\psi_n(\alpha_1)=\cdots=\psi_n(\alpha_{n-1})=0;\quad \psi_n(\alpha_n)=1.$$

This immediately leads to expansions of the form

$$u(z)\sim\sum_{n=0}^{\infty}a_n\psi_n(z)$$

which can be construed both as an orthonormal series and as an interpolation series. An explicit representation of $\psi_n(z)$ is obtained as the real part of the Walsh-Takenaka rational functions. The relation to kernel functions is shown and the case of multiple points discussed. *P. Davis.*

Kuramochi, Zenjiro. Relations between harmonic dimensions. Proc. Japan Acad. **30**, 576–580 (1954).

Let F denote a non-compact Riemann surface; let D denote a region of F which is not relatively compact and which has a frontier consisting of a countable set of analytic curves clustering nowhere in F. Let $\{F_n\}$ denote an exhaustion of F. It is assumed that F_0 is a compact disk having no point in common with D. The author considers positive harmonic functions S in D which vanish on fr D and satisfy $\int_{\mathrm{fr}\,D}\partial S/\partial n\,ds<+\infty$. With dim D denoting the maximum number of linearly independent S and dim $(F-F_0)$ denoting the maximum number of linearly independent positive harmonic functions in $F-F_0$ which vanish continuously on fr $(F-F_0)$, it is shown that dim $D\leqq$dim $(F-F_0)$. It is also shown that, if D_1, D_2 are admitted disjoint D, then dim D_1+dim $D_2\leqq$dim $(F-F_0)$. *M. Heins.*

Nikol'skiĭ, S. M. On the Dirichlet problem for the circle and half-space. Mat. Sb. N.S. **35(77)**, 247–266 (1954). (Russian)

Proofs using Fourier series methods are given for the following results. Theorem 1: If u is a harmonic function having a finite Dirichlet integral $D[u]$ on the unit circle σ, then the radial limit $\varphi(\vartheta)=\lim_{\rho\to 1-} u(\rho e^{i\vartheta})$, which exists for almost all ϑ, is in L^2 on $[0, 2\pi]$ and satisfies

$$(*) \qquad \int_0^{2\pi} |\varphi(\vartheta+h)-\varphi(\vartheta)|^2 d\vartheta \leqq M^2|h|^{1+\epsilon}$$

for $M^2=2D[u]$ and $\epsilon=0$. Theorem 2: If φ is in L^2 on $[0, 2\pi]$ and satisfies (*) for some M, $\epsilon>0$, then there exists a harmonic function u on σ having finite Dirichlet integral $D[u]<M^2\pi^{1+2\epsilon}/(2^{2\epsilon}-1)$ and having radial limit φ almost everywhere. The Dirichlet integral for half-spaces in n dimensions is studied, and similar conditions connecting the behavior of the boundary function with finiteness of the Dirichlet integral are found. *M. G. Arsove.*

Deny, Jacques. Sur la convergence de certaines intégrales de la théorie du potentiel. Arch. Math. **5**, 367–370 (1954).

The author studies integrals of the form

$$(1) \qquad F(x)=\int \mathbf{V}(t)\cdot\nabla_x h(x-t)\,dt$$

where $h(x)$ is the elementary harmonic function in R^m and where $\mathbf{V}(x)$ is a vector such that $|\mathbf{V}|^\alpha$ is summable in R^m for $1\leqq\alpha\leqq 2$. His main result is that the set of points where (1) is not absolutely convergent is of zero exterior α-capacity. He also shows that $F(x)$, defined by (1), has the property that for each $\epsilon>0$ there exists an open set Ω, of α-capacity less than ϵ, such that the restriction of F to the complement of Ω is continuous. *M. Reade* (Paris).

Zeuli, Tino. Sui potenziali di un semplice strato e di un doppio strato sferico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88**, 113–120 (1954).

Let V be the Newtonian potential inside a sphere of radius R due to a single layer on the surface of the sphere of density μ. Let F be the harmonic function inside the sphere which assumes on the surface the values μ. The formula

$$V(\rho,\theta,\varphi)=\frac{2\pi R}{\rho^{1/2}}\int_0^\rho F(t,\theta,\varphi)t^{-1/2}dt$$

is due to T. Boggio [Rend. Circ. Mat. Palermo **22**, 220–232 (1906)]. The present author derives the analogous formula for the case of a double layer; namely,

$$V(\rho,\theta,\varphi)=-\frac{\pi}{\rho^{1/2}}\int_0^\rho F(t,\theta,\varphi)t^{-1/2}dt-2\pi F.$$

Corresponding formulas are developed for potentials in the exterior of the sphere. *J. W. Green.*

Bosson, G. Note on a paper by J. L. Griffith. J. Proc. Roy. Soc. New South Wales **88**, 12–14 (1954).

[See same J. **87**, 51–62 (1953); MR **15**, 524.] The author shows that the mean value of a sufficiently regular function $f(x, y, z)$ over the surface of a sphere of centre (x, y, z) and radius λ is

$$\frac{\sinh(\lambda\Delta^{1/2})}{\lambda\Delta^{1/2}}f(x, y, z),$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ is the 3-dimensional Laplace operator. He also shows that Whittaker's solution of the wave equation can be deduced from his result. *A. Erdélyi* (Pasadena, Calif.).

Differential Equations

Azbelev, N. V. On a sufficient condition for applicability of Čaplygin's method to equations of higher orders. Dokl. Akad. Nauk SSSR (N.S.) **99**, 493–494 (1954). (Russian)

Let y be a solution of the differential equation

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$$

with initial conditions $y^{(k)}(x_0)=y_0^k$ $(k=0, \cdots, n-1)$. The author gives a sufficient condition for the existence of an upper (lower) comparison function $z(x)$, where $z^{(k)}>y^{(k)}$ $(z^{(k)}<y^{(k)})$ over a given interval $(x_0, \bar{x})$. *W. E. Milne.*

Gorbunov, A. D. On estimates of the coordinates of solutions of systems of ordinary linear differential equations. Vestnik Moskov. Univ. **9**, no. 5, 27–31 (1954). (Russian)

This is the second of two papers by Gorbunov on the same general topic. The first [same Vestnik **7**, no. 12, 3–16 (1952)] is not available to the reviewer. In this second paper he deals with the differential system $dy/dt=L(t)y+f(t)$ in which y and $f(t)$ are n-vectors and $L(t)$ is an $n\times n$ matrix. By placing suitable hypotheses on $L(t)$, $f(t)$, and the initial values, he succeeds in obtaining a set of n functions of t which serve as bounds for the n solutions $y_1(t), y_2(t), \cdots, y_n(t)$ in the interval $0<t<\infty$. *W. E. Milne.*

Trevisan, Giorgio. Su l'equazione differenziale $y''(x)+A(x)y(x)=0$.

Rend. Sem. Mat. Univ. Padova **23**, 340–342 (1954).

This note contains very short proofs of two known theorems concerning the differential equation in the title. The first [cf. R. Caccioppoli, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **11**, 251–254 (1930)] affirms that if $A(x)$ is of bounded variation and $A(x)\geqq A_0>0$ for $x\geqq x_0$, then $|y(x)|$ and $|y'(x)|$ remain bounded, as $x\to+\infty$. The second [cf. H. Milloux, Prace Mat.-Fiz. **41**, 39–54 (1934)] states that if $A(x)$ is non-decreasing and $\lim_{x\to\infty} A(x)=\infty$ then there exists a solution tending to zero as $x\to+\infty$. *W. Wasow* (Rome).

Bompiani, Enrico. Sul significato topologico di certe equazioni differenziali del secondo ordine. Rend. Sem. Mat. Univ. Padova **23**, 352–356 (1954).

The author considers the second-order differential equation (1) $v'P_i(v')=P_{i+3}(v')$, where $P_i(v')$ and $P_{i+3}(v')$ are polynomials in $v'=dv/du$ with coefficients which are functions of u and v. The interest of equations of type (1) arises from the fact that if u, v are interpreted as coordinates on a two-dimensional manifold, the type conserves its form for arbitrary regular transformations of the variables. It is shown that, if $i>0$, then equation (1) and a point determine (up to a neighborhood of second order of the point) a system of projective coordinates for which the second-order elements of the integral curves going through the point have the form $v=ku+\mu(k)u^3+\cdots$, where $\mu(k)$ is given explicitly. A projective model of the ∞^1 second-order elements through a point is given. *A. Fialkow* (Brooklyn, N. Y.).

Hartman, Philip, and Wintner, Aurel. Asymptotic integrations of linear differential equations. Amer. J. Math. 77, 45–86; errata, 404 (1955).

The linear system (1): $y'=[J+G(t)]y$ is considered in which y is a column vector of d components, J a d by d constant matrix and $G(t)$ a d by d matrix of continuous functions $g_{ik}(t)$ for $0\leqq t<\infty$. Denote the characteristic numbers of J by λ_i, and let $\mu_i=\operatorname{Re}\lambda_i$, $|G(t)|=\sum\sum|g_{ik}(t)|$. The discussion is divided into three main parts.

In Part I the asymptotic behavior of solutions of (1) is investigated under the hypothesis that $G(t)$ satisfies

$$\sup_{0\leqq V<\infty}\int_U^{U+V}\frac{|G(t)|\,dt}{(1+V)}\to 0 \quad \text{as} \quad U\to\infty. \tag{2}$$

In the second-order case the conclusions of the main theorem of this section reduce to the statement that there exists a solution $y=(y_1, y_2)$ of (1) such that $\lim_{t\to\infty}t^{-1}\log|y_m(t)|=\mu_m$, $m=1, 2$. The proofs of the results of this section are based upon the use of Riccati equations as majorants.

In Part II a refinement of a theorem of Dunkel [Proc. Amer. Acad. Arts Sci. 38, 339–370 (1902)] is proved using the results of Part I. From this theorem the following result may be established. Let $P(t)$ be a continuous d by d periodic matrix of period 1, and $F(t)$ be continuous on $0\leqq t<\infty$ with $\int_0^\infty t^{(d-1)}|F(t)|\,dt<\infty$. Let $x=x_0(t)\not\equiv 0$ be a solution of $x'=P(t)x$ and $\lim_{t\to\infty}t^{-1}\log|x_0(t)|=\mu$. Then

$$x'=[P(t)+F(t)]x$$

has a solution $x=x(t)$ satisfying $x(t)-x_0(t)=o(e^{\mu t})$ as $t\to\infty$.

Various types of results are given in Part III. For example, if for (1) there is a λ_m such that $\lambda_m\neq\lambda_j$, $j\neq m$, and $G(t)$ is of class L^p for some p, $1\leqq p\leqq 2$, then (1) has a solution $y=(y_1, \cdots, y_d)$ such that as $t\to\infty$,

$$y_m(t)\sim\exp\int^t(\lambda_m+g_{mm})du,$$

$$y_j(t)=o(|y_m(t)|) \quad \text{if} \quad j\neq m.$$

If in the scalar equation $x''-[\lambda+\varphi(t)]x=0$, $\varphi(t)$ satisfies (2) and $\operatorname{Re}\lambda^{1/2}\neq 0$, then there exists a pair of solutions of this equation which do not vanish for large t and which satisfy $x'/x\to\pm\lambda^{1/2}$ as $t\to\infty$.

For references to similar results not cited by the authors the reader should consult Bellman's recent book [Stability theory of differential equations, McGraw-Hill, New York, 1953; MR 15, 794]. *N. D. Kazarinoff* (Lafayette, Ind.).

Suyama, Yukio. On the zeros of solutions of second order linear differential equation. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. 8, 201–205 (1954).

The author considers the differential equation (1) $\ddot{y}+(\lambda-q(t))y=0$ and shows that if $q(t)$ and $q'(t)$ are continuous on the interval $0\leqq t<\infty$, if $q(t)$ increases monotonely with t, if $q(0)=0$ and $q(t)\to\infty$, as $t\to\infty$, then the number of zeros $N(\lambda)$ of a solution $y(t, \lambda)$ of (1) is given by $\pi^{-1}\int_0^{\varphi(\lambda)}[\lambda-q(t)]^{1/2}\,dt+O(1)$ for large $\varphi(\lambda)$, where $q(p(\lambda))=\lambda$. *W. Leighton* (Pittsburgh, Pa.).

Maslov, V. P. On the limit behavior of certain quantum-mechanical quantities. Dokl. Akad. Nauk SSSR (N.S.) 94, 623–626 (1954). (Russian)

The equation $h^2y''+(\lambda-u(x))y=0$ $(2m=1)$ is considered with $u(\infty)=u(-\infty)=\infty$ and $u(x)$ having a finite number of maxima and minima. Let λ_n^i be an eigenvalue and $y_n(x)$ a normalized eigenfunction on $(-\infty, \infty)$. Let $x_1, \cdots, x_{2k}$ be roots of $\lambda_n^i-u(x)=0$. Then λ_n^i satisfies one of the equations

$$h^{-1}\int_{x_{2j-1}}^{x_{2j}}(\lambda_n^i-u(x))^{1/2}dx=\pi(n+\tfrac{1}{2})+O(h)$$

for $n=1, 2, \cdots$ and $j=1, \cdots, 2k$. If the integer i is chosen so that the above equation with $j=i$ is satisfied for small h then $y_n(x)$ corresponding to λ_n^i tends to zero exponentially fast on the interval $x_{2i-1}-\alpha\leqq x\leqq x_{2i}+2$ as $h\to 0$, where α is small but not dependent on h. Other results are given.

N. Levinson (Cambridge, Mass.).

***Stoker, J. J. Mathematical methods in nonlinear vibration theory.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 28–55. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

This is an expository article in which some of the well known particular cases of the differential equation

$$\ddot{x}+\phi(\dot{x})+f(x)=H(t)$$

are discussed. Although none of the material is new the article serves as a good bird's-eye view of some of the methods and results of nonlinear vibration theory. A considerable list of references accompanies the article.

C. E. Langenhop (Ames, Iowa).

Graffi, Dario. Oscillazioni non lineari. Confer. Sem. Mat. Univ. Bari no. 1, 18 pp. (1954).

This is an expository lecture dealing with familiar parts of the theory of nonlinear oscillations. *L. A. MacColl.*

Malkin, I. G. On almost periodic oscillations of nonlinear nonautonomous systems. Prikl. Mat. Meh. 18, 681–704 (1954). (Russian)

The author discusses a system (1) $\dot{x}=X(t; x)+\mu X'(t; x; \mu)$, where x, X, X' are n-vectors and μ a small parameter, in association with the generating system (2) $\dot{x}=X(t; x)$. If the functions X, X' are periodic and of period ω, and (Poincaré) if X, X' are analytic and $\varphi(t)$ is a periodic solution of (2) and its characteristic numbers are all $\neq 2\pi Ni/\omega$ (N an integer), then φ generates a unique periodic solution $\psi(t, \mu)$ of (1) which $\to\varphi$ as $\mu\to 0$. Suppose that $\varphi=\varphi(t, h)$, depends upon a parameter h; then φ is a generating solution when and only when h satisfies a certain condition (3) $P(h)=0$. Malkin [Prikl. Mat. Meh. 16, 633–646 (1949); MR 12, 100] extended this result to the case where h is an s-vector, the condition being still (3) with P likewise an s-vector.

A necessary and sufficient condition for h to define a generating solution is that (3) hold and that the Jacobian $\partial P/\partial h\neq 0$. These results are extended more or less to the case where X is periodic, but X' is almost periodic (a.p.) of the form

$$X'=X''(x; \mu)+\sum X'''(x; \mu)e^{i\omega_\nu t}$$

where the sum is finite, the generating solution being still assumed periodic. Let an a.p. solution of (2) be defined as isolated if the corresponding variation equation has no a.p. solution. It is shown that to every isolated a.p. solution φ of (2) there corresponds a unique a.p. solution ψ of (1) which $\to\varphi$ as $\mu\to 0$. An analogue a.p. property to the property for s-dimensional parameter systems is proved here also, except

that the new $\|\partial P/\partial h\|$ must have no characteristic roots with zero real part. Furthermore, the conditions are now sufficient but not necessary. *S. Lefschetz.*

***Klotter, K. Steady state vibrations in systems having arbitrary restoring and arbitrary damping forces.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 234–257. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

This is an expository article in which the author discusses some of his recent results in the application of the Ritz-Galerkin method to the study of forced vibrations of nonlinear systems. Several graphs of the response curves for different cases are presented and a list of pertinent references accompanies the article. *C. E. Langenhop.*

Minorsky, Nicola. Sur les systèmes non-linéaires à deux degrés de liberté. Univ. e Politec. Torino. Rend. Sem. Mat. 13, 59–70 (1954).

Assuming that the constants k_1^2, k_2^2, ϵ are small, the author employs the stroboscopic method to investigate the solutions of the system of differential equations

$$\ddot{x}_1+(1+k_1^2)x_1=(a-bx_1^2)\dot{x}_1+\epsilon Q_1\dot{x}_2$$
$$\ddot{x}_2+(1-k_2^2)x_2=(a-bx_2)\dot{x}_2+\epsilon Q_2\dot{x}_1.$$

Several conclusions, relating to the existence and stability of periodic and quasi-periodic solutions, are reached; and these may well have value in engineering work. However, the heuristic nature of the method employed obscures the precise significance of the results, and makes the range of their validity difficult to judge. *L. A. MacColl.*

***Minorsky, N. Oscillatory systems containing inertial parameters.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 154–160. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

The author considers the problem of the existence of periodic oscillations in an electronic system containing two nonlinear elements, namely: (1) the usual vacuum tube, and (2) an element whose resistance depends upon its temperature, the temperature being affected by the current flowing through the element. If this problem were treated in a general and rigorous manner, it would be of considerable interest and importance. However, the present discussion involves so many uncriticized mathematical and physical approximations that it is impossible to regard any of the conclusions as having been clearly established. The paper had best be regarded as outlining a more or less plausible method of attack on the problem. *L. A. MacColl.*

Szablewski, Witold. Einfluss der Coulombschen Reibung auf Schwingungsvorgänge. Math. Nachr. 12, 183–207 (1954).

This paper deals with periodic solutions of the equation $\ddot{x}+2D\dot{x}+x+\mu \operatorname{sgn}\dot{x}=\Phi(t)$, where D and μ are positive constants, and $\Phi(t)$ is a periodic function. Such solutions can be obtained in an obvious way by solving certain inhomogeneous linear differential equations and certain transcendental finite equations. The author employs this method, with some minor modifications which are designed to facilitate the treatment of practical problems. The discussion is quite extensive, giving many details which will be useful in engineering work. *L. A. MacColl.*

Partial Differential Equations

Pólya, G., and Schiffer, M. Convexity of functionals by transplantation. With an appendix by Heinz Helfenstein. J. Analyse Math. 3, 245–346 (1954).

Diverses grandeurs "physiques" $G(D)$ attachées à un domaine D du plan ou de l'espace, telles que la rigidité à la torsion, la capacité électrostatique, etc., peuvent être définies comme l'extremum d'une fonctionnelle $I_D(f)$ ou $J_D(V)$ portant sur une fonction f (principe de Dirichlet) ou un champ de vecteurs V [principe de Thomson; voir à ce sujet: Pólya et Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951; MR 13, 270]. Soit alors D' un domaine déduit de D par un homéomorphisme simple H; si f' et V' sont les transformés par H de la fonction f et du champ de vecteurs V, supposés connus, réalisant l'extremum de I_D et J_D, les nombres $I_{D'}(f')$ et $J_{D'}(V')$ fournissent des bornes pour $G(D')$. Tel est le principe de la méthode de "transplantation" (de la fonction ou du champ de vecteurs extrémal), outil puissant pour l'obtention d'inégalités isopérimétriques (voir notamment Pólya et Szegö, loc. cit.).

Dans ce travail on considère une famille à un paramètre d'homéomorphismes H_t (qui sont essentiellement des dilatations et des représentations conformes), tels que H_1 soit l'identité, et on s'attache à l'étude de la fonction $G(D(t))$, où $D(t)$ est le transformé de $D=D(1)$ par H_t; les bornes fournies par la méthode de transplantation (exactes pour $t=1$) conduisent à des théorèmes de convexité vérifiés par $G(D(t))$, d'où le titre de l'article.

Parmi les grandeurs physiques attachées à D, on peut ranger les valeurs propres de certains problèmes de contour (du type de Dirichlet et Neumann); pour obtenir des théorèmes de convexité correspondants, les auteurs reprennent et précisent un résultat de Poincaré [Amer. J. Math. 12, 211–294 (1890)] qui donne une majoration des valeurs propres.

Les applications de la méthode paraissent innombrables; les auteurs se bornent à montrer le parti qu'on peut en tirer en traitant à fond quelques exemples, dont certains sont poussés jusqu'aux applications numériques (comme dans l'appendice, par H. Helfenstein, consacré à la rigidité à la torsion d'un triangle isocèle ou d'un rectangle).

J. Deny (Strasbourg).

Nocilla, Silvio. Sulla soluzione fondamentale di una classe di equazioni alle derivate parziali di second'ordine e di tipo ellittico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 211–224 (1954).

The equation considered is $\operatorname{div}(\rho \operatorname{grad} w)=0$, in the case of two independent variables, where ρ is a given regular function of the distance r from the origin. A fundamental solution is determined, i.e. a solution regular in an appropriate domain, containing the origin, with the exception of a critical point P', where it has a logarithmic singularity. By introducing polar coördinates, the author obtains the required solution in form of a Fourier series. In the particular case $\rho=\rho_0 \exp(\gamma_0 r^2)$, where ρ_0 and γ_0 are constants, expanding the coefficients in the Fourier series in power series in γ_0 leads to a simple approximate form. *H. Bremekamp.*

Nirenberg, L. On a generalization of quasi-conformal mappings and its application to elliptic partial differential equations. Contributions to the theory of partial differential equations, pp. 95–100. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Cet article donne des résultats très précis sur la régularité des solutions d'équations elliptiques non linéaires. Un outil essentiel est la théorie des transformations quasi conformes générales. Le résultat principal est Théorème 1: Soit z une fonction deux fois continûment différentiable dans un domaine D de R^n (coordonnées $x_1, x_2, \cdots, x_n$), solution de l'équation elliptique

$$(1) \qquad F(x_1, \cdots, x_n, z, z_1, \cdots, z_n, z_{11}, \cdots, z_{nn})=0,$$

où $z_i=\partial z/\partial x_i$, $z_{ij}=\partial^2 z/\partial x_i \partial x_j$, et où F est une fois continûment dérivable par rapport à $x_1, \cdots, z_{nn}$. Alors les z_{ij} sont continues höldériennes dans D.

Comme conséquence on peut appliquer les théorèmes de Giraud [Ann. Sci. Ecole Norm. Sup. (3) 47, 197–266 (1930)] et Hopf [Math. Z. 34, 194–233 (1931)] (auteurs qui supposent à priori les z_{ij} hölderiennes). Par exemple: si F est analytique, alors z l'est.

La démonstration détaillée du théorème 1 est réservée pour un autre article; seules sont données les grandes lignes, dont voici un apercu: a) Soit z solution de (1) dans D. On considère pour $h>0$ assez petit la fonction z^h donnée par $z^h(x)=[z(x_1+h, x_2, \cdots, x_n)-z(x_1, \cdots, x_n)]/h$. Alors z^h vérifie

$$(2) \qquad \sum_{i,j} a_{ij} z^h_{ij}=f,$$

les fonctions a_{ij} et f étant données, dépendant de h (et z); $|f|$ est uniformément bornée lorsque h varie; (2) est uniformément elliptique, au sens suivant: pour tout système de nombres réels ξ_i, on a

$$m\sum_i \xi_i^2 \leqq \sum_{i,j} a_{ij}\xi_i\xi_j \leqq M\sum_i \xi_i^2,$$

m et $M>0$, indépendants de h. b) On en déduit, par utilisation des transformations quasi conformes généralisées, que les fonctions z_i^h sont uniformément höldériennes, de sorte que l'on peut faire tendre h vers 0 et obtenir le théorème. Une transformation $x \to (p_1(x), \cdots, p_n(x))$ est dite quasi conforme généralisée si

$$\sum_{i,j} p^2_{ij} \leqq k\sum_{i,j}(p_{ij}p_{ji}-p_{ii}p_{jj})+k_1,$$

avec $p_{ij}=\partial p_i/\partial x_j$, $0<k<(n-1)/(n-2)$, $k_1 \geqq 0$.

Ces transformations ont également des propriétés de type Liouville, d'où l'on déduit: soit u solution de $\sum_{i,j} a_{ij}u_{ij}=0$ dans R^n, avec $\sup \sum_{ij}(a_{ij}-\delta_{ij})^2<1/(n-1)$. Si parmi les fonctions $\partial u/\partial x_i$, $n-1$ d'entre elles sont bornées, alors u est linéaire. *J. L. Lions* (Nancy).

Finn, Robert. On equations of minimal surface type. Ann. of Math. (2) 60, 397–416 (1954).

The author studies certain classes of partial differential equations of the form

$$(1) \quad a(p, q)r+2b(p, q)s+c(p, q)t=0 \qquad (p=\varphi_x, q=\varphi_y, r=\varphi_{xx}, s=\varphi_{xy}, t=\varphi_{yy}).$$

The coefficients a, b, c are supposed to be defined in the whole (p, q)-plane, $a>0$, $c>0$ and $ac-b^2=1$. It is further assumed that they are Hölder-continuous in every bounded part of the (p, q)-plane.

Each class of equations investigated is characterized by a property expressing a bound for the deviation of the equations of the class from the minimum-surface equation, for which the coefficients are

$$a_1=\frac{1+q^2}{W}, \quad b_1=\frac{pq}{W}, \quad c_1=\frac{1+p^2}{W}$$

with $W=(1+p^2+q^2)^{1/2}$. The class $E_1(\epsilon)$ contains the equations for which

$$(2) \qquad ac_1+ca_1-2bb_1 \leqq 2\epsilon.$$

The constant $\epsilon \geqq 1$, and only the minimal-surface equation lies in $E_1(1)$. According to L. Bers each equation (1) can be put into the form

$$(3) \qquad \frac{\partial \Theta(p, q)}{\partial x}+\frac{\partial \Lambda(x, y)}{\partial x}=0$$

in the whole (p, q)-plane. The class $E_2(\epsilon)$ consists of those equations of $E_1(\epsilon)$ for which Θ and Λ can be chosen in such a way that

$$(4) \qquad (p\Theta+q\Lambda)/(\Theta^2+\Lambda^2) \geqq \text{const.} >0.$$

Finally, a differential equation of $E_3(\epsilon)$ is an equation of $E_1(\epsilon)$ for which Θ and Λ can be chosen so that

$$(5) \qquad \Theta^2+\Lambda^2<\text{const.}$$

It is shown that $E_3(\epsilon) \subset E_2(\epsilon)$.

The new results of the paper can be grouped in the following way. The first group gives bounds for the absolute value of the gradient of a solution $\varphi(x, y)$ of one of the classes either in terms of the area of the surface $z=\varphi(x, y)$ over the domain D of the (x, y)-plane in which $\varphi(x, y)$ is defined or in terms of a bound for $|\varphi(x, y)|$ in D. The second group considers growth theorems which may be considered as analogues of Schwarz's lemma in function theory. A third group gives approximation theorems giving bounds for the deviation of two solutions of the same equation in terms of their deviation on the boundary or bounds for the deviation of two solutions of two neighboring solutions with the same boundary values. The last group finally gives fundamental existence theorems for the first boundary-value problem. It is shown that for equations of class $E_2(\epsilon)$ a solution exists for each continuous boundary function if the domain D is bounded by a twice continuously differentiable Jordan curve with positive curvature. It is further proved that for equations of class $E_3(\epsilon)$ the assumption of convexity of the boundary curve is already sufficient. These results are remarkable because they do not hold in general for equations of form (1). Even in case of an analytic boundary curve with positive curvature the continuity of the boundary function does not guarantee the existence of a solution, as is shown by an example. Further conditions, as e.g. the three-point condition restricting the boundary values, have, therefore, to be added to assure the existence of a solution. *C. Loewner* (Stanford, Calif.).

Ezra, Jacques. Sur certains systèmes d'équations aux dérivées partielles quasi linéaires. C. R. Acad. Sci. Paris 240, 270–272 (1955).

The system is assumed reducible to the form:

$$(E) \quad \frac{\partial}{\partial x}X^i(x, y, u, v)=\frac{\partial}{\partial y}Y^i(x, y, u, v)+Z^i(x, y, u, v) \qquad (i=1, 2).$$

Let D be a domain bounded by a sufficiently smooth curve C, let $f(P, Q)$ have two continuous derivatives if $P \neq Q$ and

behave near $P=Q$ like $k(P,Q)\log PQ+h(P,Q)$ with h and k regular, and let c denote a small circle with center at Q. Starting with $\int_{C-c} X^i\,df$, the author passes to a double integral, uses (E), integrates by parts, lets the radius of c approach 0, uses the principal value of improper integrals, and arrives at formulas

$$(F)\quad -\pi k(Q,Q)Y^i(Q,u(Q),v(Q)) = \int_D (f_{xy}Y^i - f_{xy}X^i - f_yZ^i)\,dxdy + \int_C f_y(Y^i\,dx + X^i\,dy),$$

which do not involve derivatives of u or v. Conditions are indicated under which (F) imply (E). Applications include the Dirichlet problem and the Tricomi equation.

F. A. Ficken (Knoxville, Tenn.).

*Douglis, A. **A geometric treatment of linear hyperbolic equations of second order.** Contributions to the theory of partial differential equations, pp. 231–234. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Bref exposé de la méthode de l'auteur [cf. Comm. Pure Appl. Math. **7**, 271–295 (1954); MR **16**, 44] pour résoudre les équations linéaires hyperboliques du second ordre au moyen de coordonnées liées aux conoïdes caractéristiques et de la formule de Green. L'auteur annonce la démonstration à l'aide de cette méthode de l'hypothèse d'Hadamard sur le principe de Huyghens. *Y. Fourès-Bruhat.*

Petrovskiĭ, I. G., and Čudov, L. A. On the curves and two-dimensional surfaces along which a solution of the wave equation can have a discontinuity. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 175–180 (1954). (Russian)

The topic of the title is elucidated for plane waves. For singularities along a line in the (x, y, z, t)-space continuity of all the third derivatives near such a line is impossible unless the line lies within a characteristic cone, for which case examples are constructed. Similar arguments apply to the possibility of a plane of singularities. Such singular solutions may be generalised solutions in a certain sense.

F. V. Atkinson (Oxford).

Lovass-Nagy, Viktor. Lösung, mit Hilfe des Matrizenkalküls, eines partiellen Differentialgleichungssystems, das zur Beschreibung der elektrischen Eigenshaften in einem aus parallelleitern bestehenden System dient. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 457–466 (1954). (Hungarian. Russian and German summaries)

The system of differential equations for current and voltage in $n+1$ parallel conductors leads to a matrix differential equation which is a straightforward generalization of the telegraphist's equation. This matrix differential equation has "plane wave" solutions, and these solutions are written out explicitly in the case that n conductors are identical and arranged in a regular pattern on the surface of a cylinder, the $(n+1)$st conductor being the axis of the cylinder. *A. Erdélyi* (Pasadena, Calif.).

Hodžaev, L. Š. Cauchy's problem for a linear parabolic equation in functional space. Dokl. Akad. Nauk SSSR (N.S.) **99**, 31–33 (1954). (Russian)

Sufficient conditions are given for the existence and uniqueness of a solution of Cauchy's problem for a differential equation of the type referred to in the title. The definition of 'solution' and the method of attack are similar to those of Gel'fand and Šilov [Uspehi Mat. Nauk (N.S.) **8**, no. 6(58), 3–54 (1953); MR **15**, 867]. *E. Hewitt.*

Ray, Daniel. On spectra of second-order differential operators. Trans. Amer. Math. Soc. **77**, 299–321 (1954).

By generalizing Rosenblatt's [same Trans. **71**, 120–135 (1951); MR **13**, 258] probabilistic construction of the Green's function $K(x, y, t)$ of the differential equation

$$\tfrac{1}{2}\Delta\psi - V(x)\psi = \partial\psi/\partial t \quad (t>0),$$

the author gives, by Laplace transform

$$\int_0^\infty \exp(-st)K(x,y,t)\,dt,$$

the explicit Green's function of $L_0=\frac{1}{2}\Delta-(s+V(x))$ in a certain open subset Ω of R^N with boundary value zero. The result is applied to derive the asymptotic behaviour (Carleman, Weyl) of the orthonormal eigenfunctions $\{\phi_n(x)\}$ and the corresponding eigenvalues $\{\lambda_n\}$ of L_0:

$$\text{(i)}\quad \sum_{\lambda_n<\lambda} \phi_n(x)^2 \sim (\lambda/2\pi)^{N/2}/\Gamma(\tfrac{1}{2}N+1)$$

$$\text{(ii)}\quad \sum_{\lambda_n<\lambda} 1 \sim (\lambda/2\pi)^{N/2}m(\Omega)/\Gamma(\tfrac{1}{2}N+1)$$

in case Ω is bounded. It is shown that (i) holds good in the case of unbounded Ω if $\lim_{|x|\to\infty} V(x)=\infty$. An extension of (ii) for unbounded Ω is also derived under an additional condition concerning the growth of $V(x)$. *K. Yosida.*

Pini, Bruno. Maggioranti e minoranti delle soluzioni delle equazioni paraboliche. Ann. Mat. Pura Appl. (4) **37**, 249–264 (1954).

If the functions $\chi_1(y)<\chi_2(y)$, $a\leqq y\leqq b$, are continuous together with their first derivatives, the set D of points (x, y) such that $\chi_1(y)<x<\chi_2(y)$, $a<y<b$, will be called a normal domain. The boundary of D exclusive of the points $y=a$, $\chi_1(a)<x<\chi_2(a)$, will be denoted by S. Let $v(x, y)$ be a continuous function defined over a region A, and let D be any normal domain contained in A. The function $u=u(x, y; D)$ which in D satisfies the equation

$$\mathfrak{M}(u)=u_{xx}+u_y=0$$

and on S equals $v(x, y)$ is said to be the function associated with v relative to the normal region D. The continuous function $v(x, y)$ is said to be $\mathfrak{M}$-prevalent [$\mathfrak{M}$-subvalent] in A if in each normal domain D of A the following relation holds between v and its associated function u

$$v(x,y)\geqq u(x,y;D)\quad [v(x,y)\leqq u(x,y;D)].$$

$\mathfrak{M}$-prevalent and $\mathfrak{M}$-subvalent functions play for $\mathfrak{M}(u)=0$ roles analogous to the roles played by superharmonic and subharmonic functions relative to the equation $u_{xx}+u_{yy}=0$. In particular, if v belongs to class C^2 and is $\mathfrak{M}$-prevalent in A, then $\mathfrak{M}(v)\leqq 0$. The main objective of the paper is to give necessary and sufficient conditions for a function v of class C to be $\mathfrak{M}$-prevalent [$\mathfrak{M}$-subvalent] in a region A. In the last part of the paper the equation $u_{xxxx}+2u_{xxyy}+u_{yy}=0$ is treated along similar lines. *F. G. Dressel.*

Zeuli, Tino. Sul problema delle temperature variabili in una calotta sferica sottile. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 335–356 (1954).

Let θ and ψ respectively be used to denote the colatitude and the longitude of points on a sphere of radius R. The

cap $0\leq\theta\leq\theta_0\leq\frac{1}{2}\pi$, $0\leq\psi<2\pi$ of the sphere will be symbolized by $C(\theta_0)$, and $FC(\theta_0)$ will denote the circle $\theta=\theta_0$, $0\leq\psi<2\pi$ bounding the cap $C(\theta_0)$. The author first shows that if the temperature u of a curved surface, submerged in a medium of constant temperature zero, is initially zero, and if the boundary of the surface is kept at zero temperature, then u is identically zero. He then finds the temperature $u=u(\theta,\psi,t)$ of the cap $C(\theta_0)$ with preassigned continuous temperature on $FC(\theta_0)$ and continuous initial temperature on $C(\theta_0)$. The solution u is expressed as an infinite series of orthogonal functions. Mathematically the problem is to find a function u which for $t>0$ satisfies in $C(\theta_0)$ the equation

$$(\sin\theta U_\theta)_\theta/\sin\theta+U_{\psi\psi}/\sin^2\theta=\alpha u+\beta+\gamma u_t,$$

and is such that

$$U(\theta_0,\psi,t)=g(\psi,t) \text{ on } FC(\theta_0)$$

and

$$\lim_{t\to 0} U=f(\theta,\psi) \text{ on } C(\theta_0).$$

Here $\alpha>0$, β, $\gamma>0$ are constants, and g and f are given functions. *F. G. Dressel* (Durham, N. C.).

Baratta, Maria Antonietta. Sopra un problema di ripartizione del calore. Riv. Mat. Univ. Parma **5**, 197–207 (1954).

Let R_1 and R_2 be the surfaces of two coaxial cylinders of radii $0<r_1<r_2$ and let S_1 denote the region interior to R_1 while S_2 denotes the region between R_1 and R_2. Assume $f_i(r,\theta)$, $U'(\theta,t)$ are given functions of class C^2 which are of period 2π in the argument θ. $L(u)$ and $B_i(u)$ denote respectively the operators $a^2(u_{rr}+u_r/r+u_{\theta\theta}/r^2)-u_t$ and u_r+k_iu, where a and k_i are constants. The author obtains series representations to solutions $u_1=u_1(r,\theta,t)$ and $u_2=u_2(r,\theta,t)$ of the following problems. (1) In S_1 the function u_1 satisfies for $t>0$ the heat equation $L(u_1)=0$, while $B_1(u_1)=k_1U'$ on R_1, and for $t=0$ and $0\leq r<r_1$ one has $u_1=f_1$. (2) For $t>0$ the function u_2 satisfies $L(u_2)=0$ in S_2 and $B_1(u_2)=k_1U'$ on R_1, while $B_1(u_2)=0$ on R_2, in addition for $t=0$ and $r_1<r<r_2$ the function u_2 meets the condition $u_2=f_2$.

F. G. Dressel (Durham, N. C.).

Shibata, Takashi. System of differential equations which are equivalent to Dirac's equation for hydrogen atom. J. Sci. Hiroshima Univ. Ser. A. **17**, 371–376 (1954).

The author obtains sixteen scalar equations $\partial\psi/\partial x^k=m_k\psi$ $(k=1,2,3,4)$ for the components of a 4-vector ψ; here $x^k=(r,\theta,\varphi,t)$ and M_k are 4×4 matrices depending on the azimuthal and magnetic quantum numbers. Solutions ψ are displayed in terms of spherical harmonics and Dirac's radial functions. These expressions satisfy Dirac's four scalar equations for the hydrogen atom. *F. A. Ficken.*

Dorfman, A. G. Investigation of possibilities of varying the solutions of certain classes of differential equations. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 167–174 (1954). (Russian)

Problems which arose originally for the deformation equation [see Efimov, Trudy Mat. Inst. Steklov. **30** (1949); MR **12**, 531] are discussed here for the following simpler equations:

$$(*)\qquad \begin{aligned} f_{uu}z_{vv}-2f_{uv}z_{uv}+f_{vv}z_{uu}&=0\\ (z_{uu}z_{vv}-z^2_{uv})-(f_{uu}f_{vv}-f^2_{uv})&=0\\ 2(z_{uu}z_{vv}-z^2_{uv})-(f_{uu}z_{vv}-2f_{uv}z_{uv}+f_{vv}z_{uu})&=0, \end{aligned}$$

of which (*) is geometrically interesting as the equation for infinitesimal (rigid) deformation. Stability, or relative stability of order l, of a solution of one of these equations are defined in exactly the same way as rigidity, or relative rigidity of order l, in case of the deformation equation [see Efimov, loc. cit.]. Consider a solution $f(u,v)$ of the form $f(u,v)=f^{(n)}(u,v)+f^{(n+1)}(u,v)+\cdots$, where $f^{(k)}(u,v)$ is a form of degree k in u, v, and $n\geq3$. The emphasis is on the cases $n=3$, 4 which are unsolved for the deformation equation. The following results are of particular interest for differential geometry. If $f(u,v)=f^{(3)}+f^{(4)}$, where $f^{(3)}=a_1u^2v+a_2uv^2$, $f^{(4)}=a_3u^2v^2$, $a_i\neq0$, then $z=f(u,v)$ possesses at least a two-parameter family of infinitesimal deformations. (This $f(u,v)$ is a solution of the second equation above with relative stability of order 1.) For each $n\geq4$ there is a surface of the form $z=f(u,v)$ which is relatively rigid of order 1, but is not infinitesimally rigid.

H. Busemann (Paris).

Difference Equations

***Herlestam, Tore. On linear difference equations with constant coefficients.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 61–78 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author studies the difference equation

$$1)\qquad \sum_{\nu=0}^{g} a_\nu f(x+\nu)=\varphi(x)$$

for real variable x and real coefficients a_ν. Beginning with the well-known result that the general solution of this equation is obtained by adding to any particular solution the general solution of the associated homogeneous equation, the author selects a principal solution from the general manifold by a process similar to that used by Nörlund in the selection of a principal solution of the summation and mean-value problems. Eq. 1) is written in operator form $a(T)f(x)=\varphi(x)$, where $a(T)=\sum_{\nu=q}^{g}(a^{(\nu)}(1)/\nu!)(T-1)^\nu$ and $q=$ least r such that $a^{(r)}(1)\neq0$. Associated with this operator the author defines the concept of "a-periodic extension" of a function defined on the fundamental interval $x_0\leq x\leq x_0+g$, and constructs a set of Bernoulli polynomials satisfying $a(T)A_n(x)=n^{(q)}x^{n-q}$, $n\geq q$, $a(T)A_n(x)=0$, $n<q$, and an Euler summation formula corresponding to $a(T)$. By applying this Euler sum formula to the a-periodic extensions of the Bernoulli polynomials the author determines the "Eulerian solution of order m" of Eq. 1), which exists if $\varphi(x)$ satisfies certain regularity conditions involving the first m derivatives of $\varphi(x)$. This solution is unique to within additive arbitrary polynomials of degree less than q. Under certain conditions this Eulerian solution may be obtained as an m-iterated integral of a particular solution of $a(T)f(x)=\varphi^{(m)}(x)$. Further, a form of "$a$-Fourier Series" associated with $a(T)$ is developed which represents the solution if the corresponding generalized series converges.

P. E. Guenther (Cleveland, Ohio).

Chakrabarti, S. C. On higher differences. I, II. Rend. Sem. Mat. Univ. Padova **23**, 255–269, 270–276 (1954).

I. This paper is a study of two operators. The first of these is defined as follows:

$$A^rv_x=A^{r-1}v_{x+1}-a^{r-1}A^{r-1}v_x, \quad A^0v_x=v_x.$$

The second is defined in the following way:

$$A_r u_x = A_{r-1} u_{xx} - a^{r-1} A_{r-1} u_x, \quad A_0 u_x = u_x.$$

Symbols too numerous to give here are defined in terms of these two basic operators. A number of formulas are then proved, most of which are generalizations of well-known formulas involving the usual difference operator, $\Delta u_x = u_{x+1} - u_x$. Certain functions are defined by infinite series in which the symbols of the paper are used. These include generalizations of the exponential and logarithm.

II. The author's introduction is as follows: "This paper is a continuation of Note I on the subject of Higher-Differences. We make use of the same notation employed in Note I. The object of this paper is to study the relations between the operators F and the operators in differential calculus."

The operators F are defined in Note I in terms of the operators A described above. *T. Fort.*

Integral Equations

Chang, Shih-Hsun. A relation between characteristic values and singular values of linear integral equations. Sci. Sinica **3**, 237–245 (1954).

The paper shows that if the characteristic values μ_i of a kernel $K(x, y)$ in L^2, and the positive singular values λ_i (λ_i^2 characteristic values of $KK^*(x, y) = \int K(x, t)\bar{K}(y, t)\,dt$) are arranged in order: $|\mu_1| \leqq |\mu_2| \leqq \cdots$ and $\lambda_1 \leqq \lambda_2 \leqq \cdots$, then $\lambda_1 \leqq |\mu_1|$ and $\lambda_1\lambda_2 \cdots \lambda_n \leqq |\mu_1\mu_2 \cdots \mu_n|$. The procedure is an extension of that for matrices [see, e.g., H. Weyl, Proc. Nat. Acad. Sci. U. S. A. **35**, 408–411 (1949); MR **11**, 37]. Also for $\det K(s_i, t_j)$, $i=1, \cdots, n$, $j=1, \cdots, n$, there exists a decomposition into n factors $\mathfrak{K}_1 \cdots \mathfrak{K}_n$, where $\mathfrak{K}_1\mathfrak{K}_2(s, t) = \int \mathfrak{K}_1(s, u)\mathfrak{K}_2(u, t)\,du$, $s = s_1, \cdots, s_n$, $t = t_1, \cdots, t_n$. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Gahov, F. D., and Čerskiĭ, Yu. I. Integral equations of convolution type. Dokl. Akad. Nauk SSSR (N.S.) **99**, 197–199 (1954). (Russian)

The method of Fourier transforms in the complex plane is used to discuss the solutions of integral equations of the form $f(x) + \int_{-\infty}^{\infty} k(x-t)f(t)dt = g(x)$, where the range of x may be $(-\infty, \infty)$, $(0, \infty)$ or $(-\infty, 0)$, and where k and f are required to be of exponential order at infinity. Results are not given in detail, and it is difficult to say how far they would go beyond those of Wiener and Hopf and of Titchmarsh [e.g. Theory of Fourier integrals, Oxford, 1937, Ch. XI]. *J. L. B. Cooper* (Cardiff).

Lambe, C. G. Integral equations satisfied by Lamé-Wangerin functions. Proc. Edinburgh Math. Soc. (2) **10**, 71–75 (1954).

For the solutions of Lamé's equation of order n, where n is the half of an odd positive integer, the author obtains (singular) integral equations with simple kernels $(1 - zt/a)^{2n}$ and $(z-t)^{2n}$. Each of the two canonical solutions of the differential equation is expressed as a contour integral involving the other canonical solution. [See also Erdélyi et al., Higher transcendental functions, vol. III, McGraw-Hill, New York, 1955, sec. 15.6; MR **16**, 586.] *A. Erdélyi.*

Functional Analysis

Pták, Vlastimil. Weak compactness in convex topological linear spaces. Čehoslovack. Mat. Ž. **4**(79), 175–186 (1954). (Russian summary)

Let X be a complete locally convex topological linear space (over the real numbers). Let M be a subset of X that is pseudo-compact (i. e., all continuous real-valued functions on M are bounded) in the weak topology of X. Then the smallest symmetric closed convex set containing M is weakly compact. This generalizes a theorem of Eberlein [Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); MR **9**, 42; **10**, 855] and also a theorem of Kreĭn and Šmulian asserting that the closed convex hull of a weakly compact subset of a real Banach space is weakly compact [Ann. of Math. (2) **41**, 556–583 (1940); MR **1**, 335]. As lemmas, the author proves the following. A) A completely regular space T is pseudo-compact if and only if for every countable set $\{x_n\}$ of real-valued continuous (bounded) functions on T and every point $s \in \beta T$ (βT is the Stone-Čech compactification of T), there exists a point $t \in T$ such that $x_n(s) = x_n(t)$ for $n = 1, 2, 3, \cdots$. B) Let T be as in A). Let $C(\beta T)$ be the space of all continuous real-valued functions on βT with topology based on the norm $\|x\| = \max\{|x(s)|; s \in \beta T\}$. Let B be a subset of $C(\beta T)$ compact in the topology of $C(\beta T)$ based on uniform convergence on finite subsets of βT. Then B is weakly compact. [In connection with this paper, see also A. Grothendieck, Amer. J. Math. **74**, 168–186 (1952); MR **13**, 857; and J. Dieudonné, Arch. Math. **3**, 436–440 (1952); MR **14**, 1093.] *E. Hewitt* (Seattle, Wash.).

Pták, V. On a theorem of W. F. Eberlein. Studia Math. **14** (1954), 276–284 (1955).

A simple proof is given of the following theorem. Let X be a complete locally convex topological linear space, and let $A \subset X$ be weakly pseudocompact. Then the weak closure of A is weakly compact. This is a special case of a theorem proved previously by the author, but by more elaborate methods (see the preceding review). *E. Hewitt.*

Laugwitz, Detlef. Über vollständige Normtopologien in linearen Räumen. Arch. Math. **6**, 128–131 (1955).

This note completes some results of an earlier note [Arch. Math. **4**, 455–460 (1953); MR **15**, 436] by finding the number of distinct norm topologies or Hilbert-space topologies in which a vector space E of linear dimension a is a complete normed space. *M. M. Day* (Urbana, Ill.).

Campos Ferreira, Jaime. On the equivalence of norms in vector spaces. Gaz. Mat., Lisboa **15**, no. 58, 5–9 (1954). (Portuguese)

This note discusses briefly the definition of dimension in a vector space over the reals and then demonstrates that in an infinite-dimensional space there are non-equivalent norm topologies. The author seems to be unaware of papers such as that of D. Laugwitz cited in the preceding review. *M. M. Day* (Urbana, Ill.).

Brace, John Wells. Compactness in the weak topology. Math. Mag. **28**, 125–134 (1955).

This is an exposition of theorems of Šmulian and Eberlein concerning the equivalence of various types of weak compactness in normed linear spaces. Stronger results have been established by Dieudonné and Schwartz [Ann. Inst. Fourier, Grenoble **1**, 61–101 (1950); MR **12**, 417] and Grothendieck [Amer. J. Math. **74**, 168–186 (1952); MR **13**, 857]. *V. L. Klee, Jr.* (Seattle, Wash.).

Halperin, Israel, and Miller, Norman. An inequality of Steinitz and the limits of Riemann sums. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 27–29 (1954).

Consider a normed linear space V and a function f on $[0,1]$ to V. For each partition $D=\{t_i\}_0^j$ of $[0,1]$ and associated choice $E=\{\xi_i\}_1^j$ of intermediate values, let $\Delta(D)=\max_{1\leq i\leq j}(t_i-t_{i-1})$ and let $\sum(D,E,f)$ be the Riemann sum $\sum_{i=1}^{j} f(\xi_i)(t_i-t_{i-1})$. Let Sf denote the set of all $v\in V$ such that $\|\sum(D_n,E_n,f)-v\|\to 0$ for some sequence of Riemann sums having $\Delta(D_n)\to 0$. Then Sf is a closed subset of V, and it was proved by Hartman [Quart. J. Math., Oxford Ser. **18**, 124–127 (1947); MR **9**, 137] that Sf is convex if f is bounded and V is a finite-dimensional Euclidean space. The present author obtains a sharpened form of an inequality employed by Hartman, and uses it to obtain the above result for an arbitrary Euclidean (i.e., inner product) space. In its new form, the inequality is as follows: If $\{v_1,\cdots,v_m\}$ is a subset of the Euclidean space V, P is the "parallelotope" $\{\sum_1^m t_iv_i$: for each i, $0\leq t_i\leq 1\}$, and Q is its "vertex set" $\{\sum_1^m t_iv_i$: for each i, $t_i=0$ or $t_i=1\}$, then for each $p\in P$ there exists $q\in Q$ such that $\|p-q\|\leq\frac{1}{2}(\sum_1^m\|v_i\|^2)^{1/2}$.

Reviewer's comment: It can be verified (for arbitrary V) that Sf is of density character $\leq 2^{\aleph_0}$, and that if C is a closed convex subset of V, of density character $\leq 2^{\aleph_0}$, then $C=Sf$ for some f. It would be of interest to know under what conditions Sf may fail to be convex. *V. L. Klee, Jr.*

Grothendieck, Alexandre. Résultats nouveaux dans la théorie des opérations linéaires. I. C. R. Acad. Sci. Paris **239**, 577–579 (1954).

In these notes, the author develops one of the main ideas of his thesis [see an announcement of the principal results in Ann. Inst. Fourier, Grenoble **4**, 73–112 (1954); MR **15**, 879], namely, the systematic use of the special properties of the Banach spaces of type $L^1(\mu)$, $L^2(\mu)$ or $L^\infty(\mu)$ (μ a Radon measure on a locally compact space) in order to define and study several types of linear operators which play an important part in many applications. A linear mapping of a Banach space E into a Banach space is said to be "integral" if it can be factored in three continuous linear mappings: $E\to L^\infty(\mu)\to L^1(\mu)\to F''$, where μ is a bounded measure, $L^\infty(\mu)\to L^1(\mu)$ the identity mapping, and F'' the bidual of F. A linear mapping $E\to F$ is "left (resp. right) semi-integral" if for every continuous linear mapping $L^1(\mu)\to E$ (resp., $F\to L^\infty(\mu)$), the mapping $L^1(\mu)\to E\to F$ (resp., $E\to F\to L^\infty(\mu)$) is integral. Finally, $E\to F$ is "preintegral" if for any two continuous linear mappings $L^1(\mu)\to E$, $F\to L^\infty(\mu')$, the mapping $L^1(\mu)\to E\to F\to L^\infty(\mu')$ is integral. When E and F are Hilbert spaces, the semi-integral mappings from E into F are exactly the Hilbert-Schmidt operators.

The main result announced in this note is that the identity mapping $H\to H$ of a Hilbert space is preintegral. This allows another characterization of preintegral mappings $E\to F$: they are those which can be factored as $E\to L^2(\mu)\to F$. Any Hilbert-Schmidt operator $L^2(\mu)\to L^2(\mu')$ can be factored $L^2(\mu)\to L^1(\mu'')\to L^2(\mu')$, and conversely. No proofs are given. *J. Dieudonné* (Evanston, Ill.).

Grothendieck, Alexandre. Résultats nouveaux dans la théorie des opérations linéaires. II. C. R. Acad. Sci. Paris **239**, 607–609 (1954).

Various consequences of the results of the preceding note are announced. For instance, any continuous linear mapping $L^2(\mu)\to L^1(\mu')$ can be factored in $L^2(\mu)\to L^2(\mu')\to L^1(\mu')$, the second mapping being the operation of multiplication by a function of $L^2(\mu')$. A new characterization of the "nuclear spaces" introduced by the author in his thesis is also derived. Finally, locally convex topological spaces E having a topology which can be defined by a scalar product for which E is a Hilbert space are shown to be identical to those which are simultaneously isomorphic to a quotient space of a space $L^\infty(\mu)$ and to a subspace of a space $L^1(\mu')$. *J. Dieudonné* (Evanston, Ill.).

Lovaglia, A. R. Locally uniformly convex Banach spaces. Trans. Amer. Math. Soc. **78**, 225–238 (1955).

Clarkson [same Trans. **40**, 396–414 (1936)] defined uniformly convex normed linear spaces; this paper discusses some weaker versions of this property and its relation with other conditions on the space. A normed linear space B is called uniformly convex (or locally u.c.) [or weakly l.u.c.] if $\|x_n\|=1=\|y_n\|$ and $\|x_n+y_n\|\to 2$ imply $\|x_n-y_n\|\to 0$ (same when all x_n are equal to some x) [all $x_n=x$ and f_0 an element of B^* such that $f_0(x)=\|f_0\|=1$ imply $f_0(x_n-x)\to 0$]. It is shown in §1 that a l.u.c. product of l.u.c. spaces is l.u.c.; in particular, the l_2 product of l_{p_i} spaces, where $i<p_i<\infty$ is l.u.c. even though [Day, Bull. Amer. Math. Soc. **47**, 313–317 (1941); MR **2**, 221] it is u.c. if and only if the p_i are bounded away from 1 and ∞. In §2 are many theorems relating l.u.c., w.l.u.c., w*.l.u.c., and differentiability of the norm in B: for example, if B^* is l.u.c., then the norm in B is strongly differentiable. §3 gives some sufficient conditions for isomorphism with l.u.c. spaces in terms of properties possessed by bases in some spaces: in particular, l_1 is isomorphic to a l.u.c. space and c_0 is isomorphic to a w.l.u.c. space. *M. M. Day* (Urbana, Ill.).

Pinsker, A. G. On conditions of equivalence of a Banach space and an L-space. Dokl. Akad. Nauk SSSR (N.S.) **99**, 677–679 (1954). (Russian)

Fullerton [Fund. Math. **38**, 127–136 (1951); MR **14**, 57] and Heider [Proc. Amer. Math. Soc. **3**, 833–838 (1952); MR **14**, 386] have published characterizations in Banach-space terms of those Banach spaces which are equivalent to L-spaces in the sense of Kakutani [Ann. of Math. (2) **42**, 523–537 (1941); MR **2**, 318]. The present author gives yet another such characterization, which appears not to differ greatly from those referred to. The case in which no unit exists is also treated. *E. Hewitt.*

***Achieser, N. I., und Glasmann, I. M. Theorie der linearen Operatoren im Hilbert-Raum.** Akademie-Verlag, Berlin, 1954. xiii+369 pp. DM 28.00.

Translation of Ahiezer and Glazman's Teoriya lineĭnyh operatorov v gil'bertovom prostranstve [Gostehizdat, Moscow, 1950; MR **13**, 358].

Meschkowski, Herbert. Über Hilbertsche Räume mit Kernfunktion. Arch. Math. **6**, 151–156 (1955).

The author considers a Hilbert function space $\mathfrak{H}$ with elements $f(x)$ defined in a given set $\mathfrak{M}$. According to Aronszajn [Proc. Cambridge Philos. Soc. **39**, 133–153 (1943); MR **5**, 38], a function $K(x,y)$ is called kernel function of $\mathfrak{H}$, if $K(x,y)\in\mathfrak{H}$ as a function of x for every $y\in\mathfrak{M}$ and if $(f(x),K(x,y))=f(y)$ for all $f(x)\in\mathfrak{H}$. Hitherto the kernel function has primarily been studied under the special assumption that $\mathfrak{H}$ is separable. The author shows that this assumption is unnecessary by proving that the existence of the kernel function always implies separability. From this it follows that $K(x,y)$ exists and equals $\sum\varphi_\nu(x)\overline{\varphi_\nu(y)}$, where

$\{\varphi_\nu\}$ is any complete orthonormal system in $\mathfrak{H}$, if and only if the above series is convergent.

The latter part of the paper is devoted to studying unitary transformations in Hilbert spaces with a kernel function. The results obtained show a certain formal analogy with Fourier transforms. *O. Lehto* (Helsinki).

Cordes, Heinz Otto. Über die Spektralzerlegung von hypermaximalen Operatoren, die durch Separation der Variablen zerfallen. I, II. Math. Ann. **128**, 257–289 (1954), 373–411 (1955).

In this paper the author continues his previous work [Math. Ann. **125**, 401–434 (1953); Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. IIa. **1954**, 51–69; MR **14**, 1096; **15**, 967] on the spectral resolution of certain operators A defined on the direct product of two Hilbert spaces in terms of given operators on the two factor spaces, in terms of whose resolutions the author determines the resolution of A on the product space. A partial differential operator on an L space is discussed as an example. The author's precise results are too complicated to summarize here.

F. H. Brownell (Seattle, Wash.).

***Hayes, R. M. Iterative methods of solving linear problems on Hilbert space.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 71–103. National Bureau of Standards Applied Mathematics Series No. 39. U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

The paper concerns the approximation of solutions of the linear equation (*) $Ax=b$. Here A is a linear operator in a Hilbert space, b a given vector. The operator A is supposed to be the sum of a positive definite and of a completely continuous operator. Such operators are called Legendre operators by the author. Different methods for approximating the solution of equation (*) are expounded. Methods for the approximation of eigenvalues are also considered. Applications to differential and integral equations are discussed at the end of the paper. *G. Fichera.*

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On the general Fourier expansion. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **4**, 299–308 (1954).

The rudiments of the theory of complex Hilbert spaces are set forth for a certain class of Abelian groups L, and the ring R of all (presumably endomorphic) operators on L. Here L is the basic space and R takes the place of the scalar field. Enough axioms are imposed to give the Schwarz inequality and expansions in orthonormal systems. The presentation is obscure to the reviewer, and would appear to contain little that is new. *E. Hewitt.*

Pitcher, Tom. Sets of "positive" functions in H-systems. Trans. Amer. Math. Soc. **77**, 481–489 (1954).

Let G be a separable, locally compact, unimodular group and let $H=H(G)$ be the L_2 space of G under Haar measure. Let A be the set of elements $a \in H$ such that L_a and R_a, the left and right multiplications (=convolutions) by a, are everywhere defined. Put W=weak closure of family of L_a's, $a \in A$. If P is the set of a.e. non-negative functions in H then (Theorem 1.1) the unitary $U \in W$ for which $UP \subset P$ are precisely the left-translations by elements of G.

Next, let H be an L_2 space with P its subset of a.e.non-negative functions; assume that in H an involution and partial multiplication are defined satisfying the axioms of Ambrose's H-systems [same Trans. **65**, 27–48 (1949); MR **10**, 429]. Theorem 2.1 gives complicated necessary and sufficient conditions in order that for some G [which turns out to be the group of unitaries in W mapping P to itself] $H=H(G)$.

In a letter to the reviewer the author advises that the approximate identities (pg. 485 line 12) can and should be chosen to lie in P; that his condition (1) (same page) should be replaced by: $\overline{A \cap P}=P$ & $(A \cap P)(A \cap P) \subset P$; and that the ordering of the quantities $[f]$ is defined by $[f] \geqq 0$ if and only if $[f](U) \geqq 0$ for all $U \in G$.

J. G. Wendel.

Naĭmark, M. A. On a continuous analogue of Schur's lemma. Dokl. Akad. Nauk SSSR (N.S.) **98**, 185–188 (1954). (Russian)

Theorem 1: Let R be the ring of all bounded uniformly continuous functions vanishing at infinity on the locally compact space A, to the bounded operators on the complex Hilbert space H. Let R_1 be a closed *-algebra of R such that (1) for all a in A, the representation $x \to x(a)$ of R is irreducible, (2) for $a_1 \neq a_2$, there exists an element x of R_1 such that $x(a_1) \neq x(a_2)$. Then $R_1=R$.

From this follows the analogue in question, which is formally similar to a special case of the commutator decomposition theorem in the von Neumann reduction theory; in fact, this special case would suffice for the application that is made later to groups. Specifically, it states: let M be a ring with an involution, and let $x \to T_x$ be a *-representation of M into R such that (1) for each a in A, the representation $x \to T_x(a)$ of M is irreducible, (2) if $a_1 \neq a_2$, the representations $x \to T_x(a_i)$ $(i=1, 2)$ are inequivalent. Let each element of R be identified with an operator on the Hilbert space K of all square-integrable functions on A (relative to a given regular measure on A) into H in the fashion $(Sf)(a)=S(a)f(a)$, if S is in R and f is in K. Then any bounded linear operator on K commuting with all the operators T_x, $x \in M$, is of the form $f(a) \to \lambda(a)f(a)$, where λ is a complex-valued bounded measurable function on A.

Application: the analogue to the Plancherel formula for a classical complex simple Lie group G obtained by the author and others is completed by showing that the relevant mapping of a dense subspace of $L_2(G)$ is not only isometric but can be extended to a unitary transformation onto an intrinsically definable L_2-space over a variety of dual to the group. This fact has been proved earlier for unimodular locally compact groups in general, but is here given a more explicit and concrete expression. *I. E. Segal.*

Singer, I. M. Automorphisms of finite factors. Amer. J. Math. **77**, 117–133 (1955).

Soient (X, F, μ) un espace mesuré non atomique et séparable tel que $\mu(X)=1$, $\mathfrak{G}$ un groupe dénombrable de transformations de X conservant F et μ, e l'élément neutre de $\mathfrak{G}$. Moyennant certaines conditions, dont l'ergodicité de $\mathfrak{G}$, on sait alors construire (notamment grâce au premier mémoire de Murray et von Neumann sur les facteurs) un H-système factoriel $\mathfrak{H}$ à élément unité (l'espace hilbertien $\mathfrak{H}$ étant $L^2(X \times \mathfrak{G})$). Les éléments de $L^2(X \times \mathfrak{G})$ nuls hors de $X \times \{e\}$ définissent une sous-algèbre abélienne maximale $\mathfrak{A}$ de $\mathfrak{H}$. L'A. étudie le groupe $\mathfrak{S}$ des automorphismes de $\mathfrak{H}$ laissant $\mathfrak{A}$ stable, et le sous-groupe $\mathfrak{I}$ des automorphismes intérieurs. Soit $\mathfrak{K}$ l'ensemble des éléments de $\mathfrak{S}$ qui induisent l'identité sur $\mathfrak{A}$. L'A. détermine $\mathfrak{K}$ et $\mathfrak{I} \cap \mathfrak{K}$. Il montre que $\mathfrak{S}$ est produit semi-direct de $\mathfrak{K}$ et d'un sous-groupe $\mathfrak{S}_1$ qu'il explicite. Il donne des exemples où $\mathfrak{K} \neq \mathfrak{I} \cap \mathfrak{K}$, $\mathfrak{S}_1 \neq \mathfrak{I} \cap \mathfrak{S}_1$.

Le groupe $\mathfrak{S}_1 \approx \mathfrak{S}/\mathfrak{R}$ peut s'interpréter comme groupe de transformations dans X, et l'A. étudie les relations entre $\mathfrak{G}$ et $\mathfrak{S}_1$. Enfin, l'A. donne des exemples de sous-algèbres $\mathfrak{H}_1 \neq \mathfrak{H}$ de $\mathfrak{H}$ "faiblement fermées", factorielles, et telles que le seul automorphisme de $\mathfrak{H}$ laissant fixes les éléments de $\mathfrak{H}_1$ soit l'automorphisme identique. *J. Dixmier* (Dijon).

Dye, H. A. On the geometry of projections in certain operator algebras. Ann. of Math. (2) **61**, 73–89 (1955).

Soient M et N deux C^*-algèbres. L'A. appelle orthoisomorphisme une application biunivoque θ de l'ensemble M_P des projecteurs de M sur N_P qui conserve l'orthogonalité. Il adapte d'abord les procédés employés par von Neumann dans l'étude des géométries continues. Théorème 1: Soient M et N des W^*-algèbres, M n'ayant pas de composantes de type I_2; tout orthoisomorphisme de M_P sur N_P se prolonge en une application de M sur N, somme directe d'un *-isomorphisme et d'un *-antiisomorphisme. Des résultats analogues ont été obtenus par Feldman [dissertation, Univ. of Chicago, 1954]. Les méthodes employées permettent de retrouver facilement des résultats de Kadison concernant les isomorphismes d'algèbres de Jordan, ou les isomorphismes d'espaces de Banach, entre W^*-algèbres [Trans. Amer. Math. Soc. **72**, 386–399 (1952); Ann. of Math. (2) **54**, 325–338 (1951); MR **14**, 16; **13**, 256]. Théorème 2: Soient M et N deux facteurs non de type I_{2n}, ϕ un isomorphisme du groupe M_U des opérateurs unitaires de M sur N_U; alors, il existe un *-isomorphisme (linéaire ou antilinéaire) Ψ de M sur N tel que $\phi(U)=\lambda(U)\Psi(U)$ pour $U \in M_U$, λ étant un caractère de M_U. Ceci, pour les facteurs, précise un résultat antérieur de l'A. [Duke Math. J. **20**, 55–69 (1953); MR **14**, 659]. Le th. 2 se déduit du th. 1 en observant que la restriction de ϕ aux symétries définit une application θ de M_P sur N_P, et en montrant (ce qui n'est pas trivial) que θ est un orthoisomorphisme. *J. Dixmier* (Dijon).

Ogasawara, Tôzirô. Finite-dimensionality of certain Banach algebras. J. Sci. Hiroshima Univ. Ser. A. **17**, 359–364 (1954).

Let A be a Banach *-algebra in which $k\|x\|^2 \leq \|xx^*\|$ for normal elements, while $yy^*=0$ only when $y=0$. Then these are (shown to be) equivalent: (a) A is reflexive, (b) A is weakly complete, (c) the biconjugate is separable, (d) A is finite-dimensional. The proof uses the fact that the maximal abelian *-subalgebras are spectrally representable. It is also shown that the weakly continuous operators on a Hilbert space H can be characterized by a Banach-algebra property in $B(H)$ ("weak complete continuity"). *R. Arens.*

Calculus of Variations

Leighton, Walter, and Martin, Allan D. Quadratic functionals with a singular end point. Trans. Amer. Math. Soc. **78**, 98–128 (1955).

The present paper completes previous ones by Marston Morse and W. Leighton [same Trans. **40**, 252–286 (1936)] and W. Leighton, [ibid. **68**, 253–274 (1949); MR **11**, 603]. The authors consider the functional

$$J(y)\Big|_a^b = \int_a^b (ry'^2 - py^2)\,dx,$$

where $0<a<b<d$, where $r(x)$ and $p(x)$ are continuous functions of x in $0<x<d$, and the integral is a L-integral. In the first paper mentioned above $p(x)$ was supposed to have a constant sign in a neighborhood of $x=0$. This hypothesis is completely removed here. A function $y(x)$ is said to be F-admissible in $[0, b]$ if 1) $y(x)$ is continuous in $(0, b]$ and $y(b)=0$; 2) $y(x)$ is AC and y'^2 is L-integrable on each closed subinterval of $(0, b]$. In addition, if $y(x)$ is also bounded in $[0, b)$, or continuous in $[0, b]$ with $y(0)=0$, then $y(x)$ is said to be F'-, or A-admissible, respectively. The authors seek necessary and sufficient conditions in order that $\liminf J(y)|_a^b \geq 0$ as $x \to 0+$, for all $y(x)$ which are F-, or F'-, or A-admissible in $[0, b]$. Then $J(y)$ is said to possess a F-, or F'-, or A-minimum limit in $[0, b)$. The point $x=0$ is in general a singular point here. Here are some main results. I. If $[0, b)$ contains no focal point of the y-axis, then $J(y)$ possesses an F-minimum limit (variable end-point problem). II. If $[0, b)$ contains no conjugate point of $x=0$ and $\int_a^b p(x)dx = O(1)$ as $x \to 0+$, then $J(y)$ possesses an A-minimum limit (fixed end-point problem).

L. Cesari (Lafayette, Ind.).

Sigalov, A. G. Two-dimensional problems of the calculus of variations in nonparametric form transformed into parametric form. Mat. Sb. N.S. **34(76)**, 385–406 (1954). (Russian)

The author presents the details of results announced in a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **93**, 405–408 (1953); MR **15**, 443]. The remarks in this review are intended to complement those in the previous review.

The details of the definition of the class of admissible functions $f(x, y)$ are too involved to give here. They need not be continuous or even single-valued. However the author proves that an admissible function is equal almost everywhere to a function absolutely continuous in the sense of Tonelli. To an admissible function corresponds an admissible surface, and conversely. If $\alpha>1$, the nonparametric integral for the admissible function is equal to the parametric integral over the corresponding admissible surface. For all $\alpha \geq 1$ the parametric integral has a minimum. Hence for $\alpha>1$ so does the nonparametric integral. If $\alpha=1$ the two integrals are in general not equal, so that though the parametric integral has a minimizing surface the corresponding admissible function does not necessarily yield a minimum for the nonparametric problem. *J. M. Danskin.*

Simon, Herbert A. Some properties of optimal linear filters. Quart. Appl. Math. **12**, 438–440 (1955).

The mathematical problem is that of minimizing an integral of the form $F=\int_0^T \phi(y, y', \cdots, y^{(n)})\,dt$. The author gives reasons for taking the function ϕ to be a quadratic form, but shows that even in this case a straightforward application of the Euler equation leads to an unstable filter, and hence to a physically meaningless solution.

R. Bellman (Santa Monica, Calif.).

Theory of Probability

Have

***Loève, Michel. Probability theory. Foundations. Random sequences.** D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. xv+515 pp. $12.00.

This is a textbook for graduate students and a reference book for workers in probability. It is a mathematical exposition which proceeds from first principles and should be intelligible to all with a certain mathematical maturity.

Considerable effort has been expended to make the book self-contained. The amount of material covered is prodigious. To achieve this the book is limited to purely mathematical matters and contains no applications whatever of probability to physical sciences and, with one very minor exception, no applications to statistics. The various chapters contain complementary material in the form of exercises which are often themselves interesting theorems. The major subdivisions of the book are: an introductory section which carries through a discussion of chains, a section on measure theory and integration, a section on general concepts and tools of probability theory, a section on sums of independent chance variables and the central limit theorem, and a final section on dependent chance variables, with chapters on ergodic theorems and on second-order properties.

The book gives evidence of a considerable effort to make the presentation elegant, and the author is to be congratulated on the success of his effort. There are, however, places in the book where the elegance does not make for maximum readability or ease of reference. According to the preface the prerequisite for the book is calculus. The reviewer congratulates the author and his university, whose students read this book after a course in calculus; other readers may wish to verify whether they are in the same fortunate position as the author. There are a number of minor errors, a partial list of which is available from the publishers.

Altogether, this book is a very valuable addition to the literature, a work of scholarship, and an excellent medium through which mathematicians can learn probability theory.

J. Wolfowitz (Ithaca, N. Y.).

Richter, Hans. Zur Grundlegung der Wahrscheinlichkeitstheorie. V. Indirekte Wahrscheinlichkeitstheorie. Math. Ann. **128**, 305–339 (1954).

[For parts I–IV, see Math. Ann. **125**, 129–139 (1952); 223–234, 335–343; **126**, 362–374 (1953); MR **14**, 634; **15**, 634.] This, the fifth and final paper of the series, is devoted, roughly, to the problem of inductive inference or, to use the author's more specific term, indirect probability theory. The thesis is developed that an utterly objectivistic solution of the problem of inference is not possible and that the most nearly objectivistic one that is possible does not differ materially from the usual subjectivistic one. As a step in this development it is argued that no form of Cournot's principle, the ostensibly mildly subjectivistic idea that events of sufficiently high probability can be taken as practically certain, is an adequate basis for the theory of induction. The "objectivistic" idea of degree of believability and the almost identical idea of subjective probability are developed axiomatically by methods, reminiscent of Parts II–IV, that cannot easily be summarized. *L. J. Savage.*

Rényi, Alfréd. New axiomatic construction of probability. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 369–427; discussion by Ákos Császár, 427–428 (1954). (Hungarian)

Motivated by the physicists' use of non-normalizable (i.e., possibly infinite) probabilities, the author presents an axiomatic approach to probability theory (along the lines indicated by Kolmogoroff) based on the concept of conditional probability. A conditional probability field is defined to be a quadruple (H, T_1, T_2, P), where H is a set, T_1 is a σ-algebra of subsets of H, T_2 is a subclass of T_1, and P is a real-valued function on $T_1 \times T_2$ satisfying the conditions I–III below. (The value of P at (A, B) is written as $P(A|B)$; it is to be interpreted as the conditional probability of A for given B.) I. $P(A|B) \geqq 0$ and $P(B|B)=1$. II. For each fixed value of its second argument P is countably additive in its first argument. III. $P(A|B \cap C)P(B|C)=P(A \cap B|C)$.

The main purpose of the paper is to indicate how much of classical probability theory can be re-derived (in a generalized form) in the new context. Section 1 states the axioms and derives some of their elementary consequences. Sections 2 and 3 present various forms of the law of large numbers and of the central limit theorem, respectively. Section 4 discusses Markoff chains and stochastic processes. The reviewer was occasionally puzzled by the author's conventions about infinity; thus, for instance, the examples on p. 380 are not quite clear.

The comment by Császár defines a sense in which a conditional probability field may be generated by a family of (ordinary) measures and states (without proof) some theorems on the existence of such generating measures.

P. R. Halmos (Chicago, Ill.).

Steinhaus, H. Probability, verisimilitude, credibility. Zastos. Mat. **1**, 149–172 (1954). (Polish. Russian and English summaries)

A philosophical and polemical article concerned with the applications of Bayes' formula and of various alternative procedures. These alternatives, including "likelihood," "verisimilitude," (?) "fiducial probability" and "confidence interval", are covered by a single term "verisimilitude" and, with reference to Wiener's Cybernetics [Wiley, New York, 1948; MR **9**, 598], described as a "terminological trick." The author's own solution of statistical problems consists in the evaluation of a quantity called "credibility." The concept of credibility is applicable in cases where the distribution function of an observable random variable Y is known except for the value of a parameter x, known to belong to an interval I. It is essential that x is not a random variable with known distribution, so that the premises of the problem are not sufficient for the application of Bayes' formula. In these circumstances, the "credibility" $M(x<a; Y=b)$ of the assertion that $x<a$, given that Y has assumed the value b, is defined as having the value of the a posteriori probability of $x<a$, given $Y=b$, computed on the assumption that x is a random variable uniformly distributed over I. *J. Neyman* (Berkeley, Calif.).

Klamkin, Murray S. On Barbier's solution of the Buffon needle problem. Math. Mag. **28**, 135–138 (1955).

Wise, M. E. The ratio of two factorials and some fundamental probabilities. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 513–521 (1954).

An expansion for the ratio $N!/(N-n)!$ of two factorials is derived from a contour integral, and hence one for the logarithm of this ratio. The first term of the latter is $n \log M$, where $M=N-\frac{1}{2}n+\frac{1}{2}$, or, when n is an integer, the mean of the n factors. Some well-known results in probability theory are thereby obtained more easily and precisely than from Stirling's expansion for a factorial and are extended; the relation between binomial and Gaussian distributions is found to differ from the one usually given. (Author's summary.) *D. M. Sandelius* (Göteborg).

Wise, M. E. A quickly convergent expansion for cumulative hypergeometric probabilities, direct and inverse. Biometrika **41**, 317–329 (1954).

The author expresses $\binom{N}{n}P=\binom{N}{n}\sum_{r=0}^{p-1}P_{n,r}$, where $P_{n,r}$ is the coefficient of $t^r u^{n-r}$ in $(1+t)^{Np}(1+u)^{N-Np}/\binom{N}{n}$, as a double contour integral, which, after some transformations, is expanded in powers of M^{-2}, where $M=N-\frac{1}{2}n+\frac{1}{2}$. The coefficients of this series are double contour integrals, which reduce to expressions of the type $\sum_i a_i I_h(b_i, c_i)$, where the $I_h(b_i, c_i)$ are incomplete beta functions, a_i, b_i, c_i depend on n and p, and $h=\{N(1-x)-\frac{1}{2}n+\frac{1}{2}p\}/M$. Substituting in $\binom{N}{n}$ the expansion of $N!/(N-n)!$ in powers of M^{-2} given by the author [cf. the preceding review], he obtains a rapidly convergent expansion of P. He also derives a rapidly convergent expansion for x in powers of M^{-2}, the coefficients of which depend on n, p and P only. The first two terms of both expansions are obtained explicitly. Numerical illustrations are given. *D. M. Sandelius* (Göteborg).

Zelen, Marvin. Bounds on a distribution function that are functions of moments to order four. J. Res. Nat. Bur. Standards **53**, 377–381 (1954).

The author has worked out explicit formulae for the exact upper and lower bounds of a cumulative distribution function when its moments of orders 1, 2, 3, and 4 are known. These arise from the well-known Čebyšev inequalities [cf. Shohat and Tamarkin, The problem of moments, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943, p. 115; MR **5**, 5; **13**, 1138], of which the author gives in an appendix a useful general statement (for any number of given moments) in a form suitable for applications. Throughout the paper he considers together distributions on intervals of each of the forms $[a, b]$, $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$.
H. P. Mulholland (Birmingham).

Rice, S. O. Distribution of the extreme values of the sum of n sine waves phased at random. Quart. Appl. Math. **12**, 375–381 (1955).

For independent random variables $\phi_1, \cdots, \phi_n$ uniformly distributed in $[-\pi, \pi]$ let $z=\sum_{i=1}^{n}\cos\phi_i$. The author studies the probability density $q_n(z)$ of z, and the distribution functions $[1-\Psi_n(E)]$ of $|z|$ and $[1-\Phi_n(r)]$ of the resultant r of a related random walk in two dimensions with elementary walks of unit length. Using a power series expansion for $q_n(z)$, $n-2<z\leqq n$, and an approximation for $q_n(z)$ in the central portion when n is large, an attempt is made to approximate $q_n(z)$ over its entire domain. The attack is made through the random-walk problem and the results converted to apply to $q_n(z)$. *R. A. Leibler.*

Moppert, K.-F., und Grün, F. Eine Bemerkung zur Langevin-Gleichung. Experientia **10**, 481 (1954).

The authors investigate the normality of the distributions of the random variables obtained by solving a certain difference equation, but their work is vitiated by their use of the false theorem that the only function of a normally distributed random variable which is normally distributed is a linear function. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Fractions continues aléatoires. Rend. Circ. Mat. Palermo (2) **1**, 170–208 (1952).

A general survey of the probability theory of continued fractions is made. If U is a random variable distributed absolutely continuously on the interval $[0, 1]$, and if $1/U$ is represented by the continued fraction $(a_1, a_2, \cdots)$, then $x_n=(a_n, a_{n+1}, \cdots)$ is a random variable, and the asymptotic nature of $S_n=\sum_1^n f(x_j)$ and its distribution are investigated. It is shown that, for certain functions f, the sum S_n can be identified, up to a small error, with a sum of the same form in which x_n is replaced by x_n', where $x_1', x_2', \cdots$ are mutually independent, with a common distribution function. This identification is used to derive the asymptotic distribution of S_n. *J. L. Doob* (Urbana, Ill.).

Dugué, D. Sur la convergence presque complète des moyennes de variables aléatoires (théorèmes de Hsu, Robbins et Erdös). Publ. Inst. Statist. Univ. Paris **3**, 149–152 (1954).

The author proves the sufficiency part of the stated theorem using Cantelli's method of fourth moment. (On p. 151, line 2 read n^3u_n for u_n.) It is stated that he has proved the necessity part elsewhere. *K. L. Chung.*

Loève, Michel. Termes variationnels dans le modèle limite central. C. R. Acad. Sci. Paris **240**, 722–724 (1955).

For a sequence of independent and uniformly asymptotically negligible random variables various relations between the limiting properties of the "order statistics" are stated, completing the author's previous announcements [same C. R. **239**, 1585–1587 (1954); MR **16**, 494]. It is stated that these results include some results of Darling [Trans. Amer. Math. Soc. **73**, 95–107 (1952); MR **14**, 60] and Gnedenko [Ukrain. Mat. Ž. **5**, 291–298 (1953); MR **15**, 238]. *K. L. Chung* (Syracuse, N. Y.).

O'Brien, George G. The solution of some queueing problems. J. Soc. Indust. Appl. Math. **2** (1954), 133–142 (1955).

The problems considered, which arose in studies of movements of freight cars in classification yards, all consider two independent servers, each with exponential service time, a) in tandem, b) separate, and c) in simultaneous attendance. The traffic arrivals are random, with average rate a, are served in order of arrival, and form a single waiting line from which there are no defections. In a) the first server pays no attention to the second and thus causes a waiting line between servers. The author claims that the traffic passing the first server remains random and calculates the expected time in the system (sum of waiting and service time) as the sum of the same variables for each of two single server systems. For a single server with distribution function $1-e^{-Lt}$, the expected total time is $(L-a)^{-1}$; hence he gets $(L_1-a)^{-1}+(L_2-a)^{-1}$ for two servers. In contrast, if the first server waits for the second to finish, so that the waiting line is always before the first, the expected total time is $(L_1+L_2-a)[L_1L_2-a(L_1+L_2)]^{-1}$, a result not noted by the author. For b) the same distribution is assumed for each of the two servers and the problem of course is the two server case of the many-server problem, first treated in telephone trunking by A. K. Erlang. Here the author's result for the distribution of total time in the system does not agree with that obtainable from Erlang's result for the distribution of waiting time, but that for expected time does. For c) also, the individual service time distributions are assumed alike; the combined distribution may be taken as $(1-e^{-Lt})^2$. The problem is the single-server problem with this distribution, and the Laplace transform of the waiting time has been given by Pollaczek. The author proceeds in his own way, calculating the equilibrium lengths of waiting

line as if the service distribution were exponential, and then the distribution of total time in the system from these values. His value for the expected total time is $3/(2L-3a)$, while that from the Pollaczek formula is $(3-a/L)/(2L-3a)$.

J. Riordan (New York, N. Y.).

Kiefer, J., and Wolfowitz, J. On the theory of queues with many servers. Trans. Amer. Math. Soc. **78**, 1–18 (1955).

Let customers arrive at epochs $\cdots, t_i, \cdots$ (where the $u_i=t_i-t_{i-1}$ are independent and have the distribution $dG(u)$) at a queuing-point and await their turns for service, the person at the head of the queue being served by the first of s ($s\geqq 1$) servers to become free, and the service-times v_i being independent and having the distribution $dH(v)$. It is supposed that u and v have finite positive expected values. Now let V_i^j be the unexpended service-time of the customer who is being served by the jth server at the epoch of arrival of the ith customer, and let $w_i \in R^s$ be a vector whose s components are $V_i^1, V_i^2, \cdots, V_i^s$ rearranged in $\uparrow$ order. Let F_i be the distribution-function (d.f.) of w_i. The authors show that $F=\lim F_i$ ($i\uparrow\infty$) exists; that F is a d.f. if and only if $\rho=E(v)/\{sE(u)\}<1$; and that F is a solution (and when $\rho<1$ the only d.f. solution) to the integral equation,

$$(*) \qquad F(w)=\int P(E_{u,v,w})\,dG(u)dH(v).$$

Here P is the measure on R^s defined by F, and $E_{u,v,w}\subseteq R^s$ is a Borel set defined by certain inequalities linearly involving u, v and w. When $s=1$, (*) is Lindley's equation [Proc. Cambridge Philos. Soc. **48**, 277–289 (1952); MR **13**, 759]. It is also shown that if F_i^* is the d.f. of the waiting time W_i of the ith customer then $F_i^*(W)\to F(W, \infty, \infty, \cdots, \infty)$ $(i\to\infty)$, and a similar result is found for the d.f. of the queue-size Q_i at the moment of commencement of service of the ith customer.

The author's methods cannot be summarized here; skillful use is made of the ingenious device of dominating the given stochastic process by an approximating "lattice-process" (discrete time and discrete states); in this way the authors are able to use standard theorems concerning Markov chains. The problem of solving (*) is not dealt with; for explicit solutions in certain simple cases the reader is referred to F. Pollaczek [C. R. Acad. Sci. Paris **236**, 578–580 (1953); MR **14**, 772].

D. G. Kendall (Oxford).

Lévy, Paul. Le mouvement brownien. Mémor. Sci. Math., no. 126. Gauthier-Villars, Paris, 1954. 84 pp. 1200 francs.

This monograph is partly a condensation (with some sketches of proofs) of Chapters 1, 6, 7 and 8 of the author's book, Processus stochastiques et mouvement brownien [Gauthier-Villars, Paris, 1948; MR **10**, 551], and partly a summary of some recent results obtained (mostly) since 1948. In the latter category are, in order of appearance ($x(t)$ denotes the sample function of the process): the precise form of the Lipschitz condition due to Chung and Erdös (not yet published); the precise form of the law of the iterated logarithm due to Petrovsky, Kolmogorov and Feller; the lower bounds of $\max|x(t)|$ and $\max x(t)$ in $(0, T)$ due respectively to Chung and Hirsch; the theory of functionals of $x(t)$ due mainly to Kac; the author's theory of stochastic differential equations; the problem of multiple points due to Dvoretsky, Erdös and Kakutani; the Hausdorff measure of the curve due to the author; the speed of escape to infinity in dimensions $\geqq 3$ due to Dvoretsky and Erdös (the proof of this result is however incomplete around their formula (6.16) on p. 366 of Proc. 2nd Berkeley Symposium Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 353–367 [MR **13**, 852] but can be completed); the author's definition of stochastic integral and its application to the area between the curve and its chord; the characteristic function of this area. The author informs me that the last sentence in the footnote on p. 78 is incorrect.

K. L. Chung.

Bass, J. Sur les solutions aléatoires de certaines équations aux dérivées partielles liées à des problèmes d'hydrodynamique. J. Math. Pures Appl. (9) **33**, 295–327 (1954).

For a given space-time partial differential system there is the question: which part of the statistics of the instantaneous state of the system "autopropagates" itself, in the sense that the instantaneous averages defining this part completely determine their rates of change with respect to time. The author considers the equation of Burgers, $u_t+uu_x=\mu u_{xx}$, $-\infty<x<\infty$, $\mu>0$. The reviewer has shown that the full characteristic functional

$$\phi(y(x), t)=\overline{\exp\left(i\int_{-\infty}^{\infty} y(x)u(x, t)\,dx\right)}$$

satisfies a functional differential equation $\partial\phi/\partial t=L(\phi)$, where $L(\phi)$ is a functional differential operator which is determined by the Burgers equation, in other words, that the functional ϕ is "autopropagated". The author considers the more restricted characteristic functions

$$\varphi(x_1, \cdots, x_n, y_1, \cdots, y_n, t)=\overline{\exp\left(i\sum_1^n y_n u(x_n, t)\right)}$$

and shows that these functions are not autopropagated.

E. Hopf (Bloomington, Ind.).

Urbanik, K. Limit properties of homogeneous Markoff processes with a denumerable set of states. Bull. Acad. Polon. Sci. Cl. III. **2**, 371–373 (1954).

The following theorems are stated without proof. Let $P_k^n(t)$ be the probability of a transition from state k to state n in time t, for a Markov process with denumerably many states, indexed by the non-negative integers, and with initial state α. It is supposed that almost all sample functions have both one-sided limits at all parameter points. (1) k is a limiting value of a sample function ($t\to\infty$), with positive probability, if and only if $\int_0^\infty P_\alpha^k(t)\,dt=\infty$. (2) Limit sets are disjoint. (Here a set of integers is called a limit set if it, possibly together with $+\infty$, is the set of limiting values of a sample function, with positive probability.) (3) An expression is obtained for the distribution of the maximum of a sample function of a birth and death process on the infinite parameter interval.

J. L. Doob.

Flanagan, Joseph E. Coding to achieve Markov type redundancy. J. Math. Phys. **33**, 258–268 (1954).

The author studies the problem of block coding incoming text in such a manner as to avoid some of the difficulties attendant with efficient coding such as lag, large memory capacity, and large blocks. He does this by coding inefficiently but so that the resulting 0, 1 channel stochastic process is "close" to a preassigned 0, 1 discrete, stationary, Markov α process. If $\{X_n\}_P$ and $\{X_n\}_{M(\omega)}$ are processes with probability measures P and M satisfying $M\ll P$, the meas-

ure of closeness $H[\{X_n\}_P, \{X_n\}_{M(\omega)}]$ used is defined by

$$H[\{X_n\}_P, \{X_n\}_{M(\omega)}] = \lim_{s\to\infty} E_P\left[\sum_{j=1}^{\circ} P(X_{t+1}=j|X_t, \cdots, X_{t-s}) \times \log_2 \frac{M(X_{t+1}=j|X_t, \cdots, X_{t-s})}{P(X_{t+1}=j|X_t, \cdots, X_{t-s})}\right].$$

A necessary condition for the existence of β block coding incoming text which is discrete, stationary, Markov β, ergodic and close, in this sense, to a Markov α process is proved. *R. A. Leibler* (Washington, D. C.).

Heaps, H. S. The effect of a random noise background upon the detection of a random signal. Canad. J. Phys. **33**, 1–10 (1955).

An investigation of the subject of the title for the case where both noise and signal are distributed according to a Rayleigh law. *R. A. Leibler* (Washington, D. C.).

Walsh, John E. Approximate salvo kill probabilities for small and medium sized targets when cumulative damage is unimportant. J. Operations Res. Soc. Amer. **3**, 69–76 (1955).

***Steinhaus, H., unter Mitarbeit von T. Czechowski, M. Fisz, O. Lange, J. Oderfeld und W. Sadowski. Die Wahrscheinlichkeitsrechnung als Hilfsmittel zu Untersuchungen in Naturwissenschaften und Produktion.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 69–94. Deutscher Verlag der Wissenschaften, Berlin, 1954. Survey article.

Mathematical Statistics

Hammersley, J. M., and Morton, K. W. The estimation of location and scale parameters from grouped data. Biometrika **41**, 296–301 (1954).

It is assumed that an experimenter has at hand several samples from universes of unknown form which however differ from each other only by a linear transformation of the random variable z, under observation. The objective is to estimate the scale and location parameters of the universes sampled. The observations in any one sample are grouped in intervals, $z \leqq x_0$, $x_{i-1} < z \leqq x_i$ $(i=2, 3, \cdots, n)$, and $x_n < t$ so that no interval is empty. It is further assumed that though all but one of the samples are small, this one is sufficiently large so that it gives a sufficiently good representation of the cumulative distribution of its universe. Then if in one of the other samples m_i of the total of m observations do not exceed x_i, and if $p_i = m_i/m$, the values y_i are found at which the cumulative frequencies of the "known" universe are p_i. Then the x_i's are related to y_i's by the regression equation $y_i = \alpha + \beta x_i$, and standard regression analysis is used to estimate α and β, it being noted that the y_i's are not independent. Explicit directions are given for a routinized computing procedure and a detailed numerical illustration is given. The analogy between these methods and those used by E. H. Lloyd [Biometrika **39**, 88–95 (1952); MR **14**, 65] is pointed in some detail. *C. C. Craig.*

Hagstroem, K.-G. The diagonal section of a hexahedron and the Gauss distribution bell. Nordisk Mat. Tidskr. **2**, 97–100, 135–136 (1954). (Swedish. English summary)

Given a cube with a diagonal 6 units in length, the area of a plane section perpendicular to the diagonal at distance t from the center is expressed by one of three second-degree polynomials in t, accordingly as t lies in the interval $(-3, -1)$, $(-1, 1)$, or $(1, 3)$. The composite graph of the polynomials follows the normal frequency curve quite closely. The explanation of this lies in the equivalence of the function plotted with the frequency function of the sum of three random variables each rectangularly distributed on $(0, 1)$ as pointed out by the author. *C. C. Craig.*

Marakathavalli, N. The distribution of t_1 and its applications. J. Madras Univ. Sect. B. **24**, 251–272 (1954).

The statistic t_1 is the ratio of an $N(0, 1)$ random variable to the square root of an independent non-central χ^2 divided by its degrees of freedom, n. Thus t_1 differs from the usual non-central t in the respect that the non-centrality occurs in the denominator rather than in the numerator. The author derives the exact distribution of t_1 as a series in which the terms are products of frequency functions of $tn^{-1/2}$ with increasing degrees of freedom by terms of a Poisson series. He then approximates the distribution of t_1 by a distribution function of $c^{1/2}t$, where t has ν degrees of freedom, by equating second and fourth moments. (t_1 has a symmetric distribution.) Tables of ν and $c^{1/2}$ are given for values of n and the non-centrality parameter λ of the χ^2. The approximations obtained are shown to agree rather well with exact values found by the series in a number of cases. A table of approximate 5% points to t_1 is then given for values of n and λ. Finally an application of t_1 is made to the case in which two samples are each composed of drawings in a fixed proportion from two normal universes with different means but equal variances, to test the hypothesis that the means of the two compound universes are equal, the two variances being equal. He also shows how to adapt his test to the case in which the variances are unequal but have a known ratio. *C. C. Craig* (Ann Arbor, Mich.).

Cornish, E. A. The multivariate t-distribution associated with a set of normal sample deviates. Austral. J. Physics **7**, 531–542 (1954).

Let $X_1, X_2, \cdots, X_n$ be independently and identically distributed chance variables, each with a normal distribution with variance σ^2. Define $\bar{X}$ as $(X_1+\cdots+X_n)/n$, and Y_i as $X_i-\bar{X}$, for $i=1, \cdots, n-1$. Suppose s^2 is a chance variable distributed independently of $Y_1, \cdots, Y_{n-1}$, $\nu s^2/\sigma^2$ having a chi-square distribution with ν degrees of freedom. Define t_i as $(Y_i/s)\sqrt{(n/(n-1))}$, $i=1, \cdots, n-1$. The author gives expressions for the variance-covariance matrix and joint density function for $t_1, \cdots, t_{n-1}$, marginal and conditional distributions of subsets of $t_1, \cdots, t_{n-1}$, and distributions of linear and quadratic functions of $t_1, \cdots, t_{n-1}$.

L. Weiss (Charlottesville, Va.).

Koźniewska, I. The first absolute central moment for Pólya's distribution. Zastos. Mat. **1**, 206–211 (1954). (Polish. Russian and English summaries)

Pólya distribution is understood to mean the frequency function

$$p_k = \binom{n}{k} \prod_{i=0}^{k-1}(p+i\alpha) \prod_{j=0}^{n-k-1}(1-p+j\alpha) \Big/ \prod_{m=0}^{n-1}(1+m\alpha)$$

defined for $k=0, 1, 2, \cdots, n$, where $0<p<1$, $a>-1$. The author finds that the first absolute central moment of this distribution is given by $2rp_r[1-p+(n-r)a]$, where r denotes the least integer exceeding np. *J. Neyman.*

David, H. A., Hartley, H. O., and Pearson, E. S. The distribution of the ratio, in a single normal sample, of range to standard deviation. Biometrika **41**, 482–493 (1954).

Let $w=x_{\max}-x_{\min}$ and $(n-1)s^2=\sum_{i=1}^{n}(x_i-\bar{x})^2$, where $x_1, \cdots, x_n$ is a random sample from a normal population. The distribution of $u=w/s$ is approximated by selecting a curve from the Pearson system with the proper first four moments, these being determined via the independence of u and s. This distribution is compared for specific n with that of an exact alternative derivation. After examining certain non-normal populations the authors suggest that u may be useful in detecting departures from homoscedasticity and normality. *H. Teicher* (Lafayette, Ind.).

David, H. A. The distribution of range in certain non-normal populations. Biometrika **41**, 463–468 (1954).

By performing the indicated integration in the well-known formulae for the cumulative distribution function and expected value of the range of a population with density function $f(x)$ for five parent populations, the author gives exact specialized expressions for these. Numerical comparisons with normal populations are made. The statement that an estimate of standard deviation which "is unlikely to be seriously biased therefore will provide satisfactory control limits for the mean" is hardly tenable. *H. Teicher.*

Daniels, H. E. Saddlepoint approximations in statistics. Ann. Math. Statist. **25**, 631–650 (1954).

The author presents a new approach to the problem of finding approximately the distribution of a statistic in cases where the characteristic function can be found exactly. "In this paper we show that for a statistic such as the mean of a sample . . . , or the ratio of two such means, a satisfactory approximation to its probability density, when it exists, can be obtained nearly always by the method of steepest descents", which "has been largely ignored by writers on statistical theory." Suppose, first, that x is a random variable with an absolutely continuous distribution whose range is an interval (a, b), that $f(x)$ is the density and that $e^{K(T)}=\int_a^b e^{Tx}f(x)\,dx$, with convergence when T is real and in $(-c_1, c_2)$: it is assumed that $c_1\geqq 0$, $c_2\geqq 0$, $c_1+c_2>0$, and any of $-a$, b, c_1, c_2 may be ∞. Let $\bar{x}$ be the mean of n independent x's; its density

$$f_n(\bar{x})=\frac{n}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\exp\left[nK(T)-n\bar{x}T\right]dT,$$

where $-c_1<\Re(T)<c_2$ on the path of integration. The modulus of the integrand has a saddlepoint where $K'(T)=\bar{x}$: the author shows that this equation has a single real root T_0 in $(-c_1, c_2)$ for each x in (a, b) and that $K''(T_0)>0$, provided that $K'(T)\to a$ $(T\to -c_1)$ and $K'(T)\to b$ $(T\to c_2)$. The proviso is automatically satisfied except when a or b is infinite and the corresponding c_1 or c_2 is finite. By integrating along a path part of which follows the curve of steepest descent through T_0, he establishes an asymptotic expansion

$$(*)\qquad f_n(\bar{x})=g_n(\bar{x})(1+a_1n^{-1}+a_2n^{-2}+\cdots),$$

where $g_n(\bar{x})=n^{1/2}[2\pi K''(T_0)]^{-1/2}\exp\left(nK(T_0)-n\bar{x}T_0\right)$, and $a_1, a_2, \cdots$ depend on T_0, and thus on x, but not on n. Writing

$$z=(T-T_0)[K''(T_0)]^{1/2}$$

and

$$\tfrac{1}{2}[w(z)]^2=K(T_0)-\bar{x}T_0-K(T)+\bar{x}T,$$

he gets

$$a_r=\frac{1}{2^r r!}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{z}{iw(z)}\right)^{2r+1}\right]_{z=0}\qquad (r=1, 2, \cdots).$$

He calls $g_n(\bar{x})$ the saddlepoint approximation. He notes that (*) is one of a family of expansions obtainable by a method used by H. Cramér [Théorie des probabilités, partie III, Hermann, Paris, 1938, pp. 5–23]. [Remark: $g_n(\bar{x})$ is closely related to a refinement of Cramér's approximation by W. Feller, Trans. Amer. Math. Soc. **54**, 361–372 (1943), Theorem 1, p. 363, and (2.17), p. 365; MR **5**, 125. Feller gets an explicit bound for the error: it would be interesting to know whether the author's method could be made to yield this.] The author considers the effects of various possible behaviours of $f(x)$ as $x\to a$ or $x\to b$, and obtains theorems for a wide class of distributions showing that the relative error $[g_n(\bar{x})-f_n(\bar{x})]/g_n(\bar{x})$ is $O(n^{-1})$ uniformly for $a<\bar{x}<b$.

He deals also with the ratio of sample means of two (possibly dependent) random variables, and with the case when x is an integer-valued random variable. He illustrates the cases considered by examples, in some of which the saddlepoint approximation is actually proportional to the (known) exact expression. *H. P. Mulholland.*

Trickett, W. H., and Welch, B. L. On the comparison of two means: further discussion of iterative methods for calculating tables. Biometrika **41**, 361–374 (1954).

This paper gives a more extended treatment of the method used by the second author in a previous paper [Biometrika **34**, 28–35 (1947); MR **8**, 394] to derive an exact test for the equality of means in samples from two normal universes with unequal and unknown variances. Not only is there an amplication of development which throws light on the solution reached in the earlier paper but a numerical iterative method for finding significance levels is developed and fully illustrated. *C. C. Craig* (Ann Arbor, Mich.).

***Rényi, Alfred. Eine neue Methode in der Theorie der geordneten Stichproben.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 203–212. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

A statement of some of the results in Acta Math. Acad. Sci. Hungar. **4**, 191–231 (1953); MR **15**, 885.

J. Wolfowitz (Ithaca, N. Y.).

Des Raj. Truncated sampling from distributions admitting sufficient statistics. Sankhyā **14**, 169–174 (1954).

The author considers truncated and censored samples from parametric populations of the Koopman type. He shows that for these populations the information matrix for large samples can be obtained by comparatively simple computations. These computations are illustrated in the case of the one-dimensional normal population truncated from below. *W. Birnbaum* (Seattle, Wash.).

Fieller, E. C., and Hartley, H. O. Sampling with control variables. Biometrika **41**, 494–501 (1954).

Let y and x be highly correlated random variables with unknown and known distributions, respectively,

$$p_{ij}=\text{Prob}\,(y_i<y\leqq y_{i+1},\ x_j<x\leqq x_{j+1}),$$

and n = the frequency corresponding to p_{ij} in a random sample of size N, where

$$i=1(1)m,\ j=1(1)k,\ y_0=x_0=-\infty,\ y_{m+1}=x_{k+1}=+\infty.$$

As estimator of $p_{i\cdot}=\sum_j p_{ij}$ the authors suggest $\hat{p}_{i\cdot}=\sum_j p_{\cdot j}u_{ij}$, where $p_{\cdot j}=\sum_i p_{ij}$, and $u_{ij}=n_{ij}/\sum_i n_{ij}$, if $\sum_i n_{ij}>0$, and $u_{ij}=\sum_j n_{ij}/N$, if $\sum_i n_{ij}=0$. Using a simple approximation for $\sum_{r=1}^{n} r^{-1}\binom{n}{r}p^r(1-p)^{n-r}$, the authors determine approximately var $(\hat{p}_{i\cdot})$ and compare it with the variance of the estimate $\sum_j n_{ij}/N$, which is based only on information about y. A numerical illustration is given.

D. M. Sandelius (Göteborg).

Somerville, Paul N. Some problems of optimum sampling. Biometrika **41**, 420–429 (1954).

This paper deals with the problem of selecting from k given populations with means $\theta_0\geqq\theta_1\geqq\cdots\geqq\theta_k$ the one with the largest mean. It is assumed that 1) the number of observations N to be taken in the future from the selected population is known, 2) the loss due to each possible selection is known, 3) the cost of taking n observations is linear in n with known constants, and 4) that the populations have a common, known variance. An optimum sample size n per population is then computed which minimizes the maximum risk. Normality is assumed only for the sample means, $\bar{X}_{in}$.

A sufficient condition for the existence of a maximum of the loss function is given which is quite general. The symbol $[\partial p/\partial\theta_i]_{\theta_1=\theta_2=\cdots=\theta}$ (which is meaningless in the paper since the parameters θ_i are ordered) is used to denote the operation $\partial p/\partial\theta_i$ for $\theta_i\neq\theta_j$ followed by the operation of setting all $\theta_i=\theta$. An alternative is to introduce unordered parameters.

M. Sobel (Allentown, Pa.).

Stoller, David S. Univariate two-population distribution-free discrimination. J. Amer. Statist. Assoc. **49**, 770–777 (1954).

On the basis of a sample of size N having m members in one population and $N-m$ in another, a point ζ^* is selected and a future observation is classified as belonging to population I or II according as it is $>$ or $\leqq\zeta^*$. The procedure given is distribution-free and such that the (random) probability of a correct classification as well as its estimate converge stochastically to the maximum attainable probability of a correct classification. *H. Teicher* (Lafayette, Ind.).

Ronge, Felix. Die Verhältnisschätzung (ratio estimate) nach der Methode des "Veränderungsfaktors" und der "additiven Veränderungsgrösse". Mitteilungsbl. Math. Statist. **6**, 221–232 (1954).

Let X and Y stand for the sum of all measurements in a population of size N at two different times. The author compares the following two estimates of Y: $Y_1=X\bar{y}/\bar{x}$ and $Y_2=X+N(\bar{y}-\bar{x})$, where $\bar{x}$ and $\bar{y}$ are the means of corresponding samples. *G. E. Noether* (Boston, Mass.).

Sundrum, R. M. On the relation between estimating efficiency and the power of tests. Biometrika **41**, 542–544 (1954).

The author gives a condition under which the more efficient of two unbiased normally distributed estimators of a population parameter θ leads to a more powerful test of H_0: $\theta=\theta_0$ versus H_1: $\theta=\theta_1>\theta_0$. *H. Teicher.*

Hoel, Paul G. On a property of the sequential t-test. Skand. Aktuarietidskr. **37**, 19–22 (1954).

The author discusses the Wald sequential t-test described on pages 78–84 and 203–207 of Wald's "Sequential analysis" [Wiley, New York, 1947; MR **8**, 593]. In this review, all notation will be that used by Wald. The author states that the double minimax property (that is, the property of simultaneously minimizing the maximum for both $\alpha(\theta)$ and $\beta(\theta)$ in the class C of tests) claimed for the sequential t-test does not hold, and that Wald's proof of this property outlined on pages 204–207 of the reference above is not valid. Rather than explaining where Wald's proof is incorrect, the author gives the following argument to show that the double minimax property cannot hold for the sequential t-test. When the probability ratio for Wald's test is compared with the probability ratio for Arnold's test [Tables to facilitate sequential t tests, Nat. Bur. Standards Appl. Math. Ser. no. 7 (1951); MR **13**, 141], the former is greater than the latter with probability one. From this and from the structure of the tests, it follows that Wald's $\alpha(\theta)$ is greater than Arnold's $\alpha(\theta)$, with an opposite relation for $\beta(\theta)$. And from this, it follows that neither test possesses the double minimax property.

In the opinion of the reviewer, the author's argument just summarized does not prove that Wald's test fails to possess the double minimax property. For the argument uses the fact that the same decision limits (A, B) are used in both Arnold's test and Wald's test. But these decision boundaries are, of course, only approximations, and it is conceivable that by properly adjusting A and B according to which test is being used, the inequalities for $\alpha(\theta)$ and $\beta(\theta)$ that the author uses in his argument may no longer hold.

L. Weiss (Charlottesville, Va.).

Råde, Lennart. A note on a modified t-test. Skand. Aktuarietidskr. **37**, 65–70 (1954).

Let $\{X_i\}$, $i=1, 2, \cdots, n$, $n=mk$, be a random sample from a normal population (ξ, σ). Divide the sample into k groups of m each. Define the sample in the jth group by R_j, the mean range by $R_{m,k}=k^{-1}\sum R_j$. If $\xi=0$, $\sigma=1$, define $E(R_{n,1})=\alpha_n$, σ^2 of $R_{n,1}=\beta_n$, $\gamma_n=\beta_n/\alpha_n$, $\delta_n=n\gamma_n^2$, and $W_{m,k}=R_{m,k}/\alpha_m$. The author shows that $\lambda=u[n(n+u^2\delta_m)^{-1}]^{1/2}$ is approximately normal $(0, 1)$ and hence determines the approximate distribution of $u=(\bar{X}-\xi)n^{1/2}/W_{m,k}$, where λ_θ denotes the normed normal θ fractile. Comparisons are made between the exact values of u for $\theta=.95, .975, .99$ against his approximate values of u for $m=7, 8, 9$, $k=1, 3, 5, 10, 15$ and it is shown that the agreement is good except for $k=1$. The approximate power function of the u test is derived and the approximate standardized error of the two-sided u test is given in general, and is computed for the same values of m and k as before. Agreement is good except for $k=1$. The author's results are much easier to apply, although not quite so accurate as the approximation of Patnaik [Biometrika **37**, 78–87 (1950); MR **12**, 116].

L. A. Aroian (Culver City, Calif.).

Jeeves, T. A. Identification and estimation of linear manifolds in n-dimensions. Ann. Math. Statist. **25**, 714–723 (1954).

Let X and Y be n-dimensional random (row) vectors of which the latter is observable while the former is not. It is assumed that $Y=XB+U$ in which B is an $n\times n$ matrix of sure numbers and U is another n-dimensional random (row) vector which is stochastically independent of X. Of special

concern here is the case in which U has a multinormal distribution and the main question under consideration is to determine the row space S of B. The author deals with two problems: (a) whether S is determined if Y is known (identifiability), and (b) whether a consistent estimate of S can be constructed from an infinite sequence of observations on Y.

In the n-dimensional case it is necessary to clarify the interpretation to be given to identification and this is first done. Let M be the set of $n \times n$ matrices and Θ a family of subsets of M. Let Ω be the set of n-dimensional distribution functions and $\mathfrak{X}$, $\mathfrak{U}$ and $\mathfrak{Y}$ non-empty subsets of Ω associated with X, U and Y respectively. Let F_X, F_U, and F_Y be the distribution functions of X, U, and Y respectively. Finally, let ϑ be a parameter with range $\Theta_s \subset \Theta$. From this the author formulates the following definition of identifiability (his second). For any sets $\mathfrak{X}$, $\mathfrak{U}$ and Θ, the relation $L(\vartheta, F_Y)$ holds if $B \in \vartheta$ and if $Y = XB + U$ for some X and U such that $F_X \in \mathfrak{X}$ and $F_U \in \mathfrak{U}$. Further considerations lead to assuming that the matrices belonging to ϑ are row-equivalent and to regarding ϑ as a row space; that the elements of Θ_s are the sets of row-equivalent spaces corresponding to the various row spaces of dimensions; that $\Theta = \bigcup_1^n \Theta_s$; and $\mathfrak{U}$ is the set of multinormal distributions. Then the definition of identifiability is particularized for a relation of inclusion as follows: ϑ is identifiable L^* if $\delta(\vartheta) \subset \delta(\vartheta^*)$ for every $\vartheta \in \Theta_s$, and $\vartheta^* \in \Gamma(\vartheta)$. Here $\delta(\vartheta)$ is the row space of ϑ, $\Gamma(\vartheta)$ is the set of ϑ's related by L to a particular F_X, L^* is a relation L which gives rise to the relation R of inclusion, and $R(\vartheta, \vartheta^*)$ means $\delta(\vartheta) \subset \delta(\vartheta^*)$. This is the adopted definition.

The concept of non-normality of a random variable is important in this connection. Let $nn(Y)$ be the least value of d such that $Y = U + V$, where U and V are independent, V has a multinormal distribution, and U has the dimension d. Then $nn(Y)$ is the non-normality of Y. Further, $nn(\mathfrak{Y}) = S$ if $nn(Y) = S$ for every Y such that $F_Y \in \mathfrak{Y}$. Then the author's main result on identifiability is: ϑ is identifiable (L^*) if and only if $nn(\mathfrak{Y}) = s$. After deducing some consequences of this theorem, a method of constructing an estimate of the linear structure from an infinite sequence $Y_1, Y_2, \cdots$ is given and shown to be consistent. *C. C. Craig.*

Reiersøl, Olav. Tests of linear hypotheses concerning binomial experiments. Skand. Aktuarietidskr. **37**, 38–59 (1954).

The author studies a number of problems where one tests a hypothesis about the probabilities of systems of binomial chance variables. The problems include some where multiple classifications occur. He obtains minimum chi-square statistics to test these hypotheses and proves that the asymptotic distributions of these statistics are chi-square with specified numbers of degrees of freedom. It is not possible to describe the problems within present space limitations. *J. Wolfowitz* (Ithaca, N. Y.).

Jenkins, G. M. Tests of hypotheses in the linear autoregressive model. I. Null hypotheses distributions in the Yule scheme. Biometrika **41**, 405–419 (1954).

Let $p_n(r_1, \cdots, r_k)$ be the joint distribution of the first k serial coefficients of variates $x_1, \cdots, x_n$ which (Hypothesis H_0:) are independent and normal $(0, \sigma)$. Extending the small-sample methods available for $k=1$ the author obtains smoothed form of the joint characteristic function of a $p_n(r_1, r_2)$ with correct bivariate moments up to order n, a typical conclusion being that the smoothed distribution of the serial multiple correlation coefficient [m.c.c.] of x_t on x_{t+1} and x_{t+2} is the same as that of an ordinary m.c.c. based on $n+3$ observations. [Reviewer's remark: The results are of wider bearing than indicated by the titles, for H_0 may be a null hypothesis also in nonautoregressive models, e.g. in nonautoregressive moving averages.] *H. Wold* (Uppsala).

Mathematical Economics

***Decision processes. Edited by R. M. Thrall, C. H. Coombs, and R. L. Davis.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1954. viii +332 pp. $5.00.

This volume contains an introduction and 18 papers presented at a seminar held in the summer of 1952, dealing generally with the mathematical formulation and empirical testing of propositions about human behavior. After the introduction by R. L. Davis (pp. 1–18) summarizing the papers and putting them in context, there is an expository paper by C. H. Coombs, H. Raiffa, and R. M. Thrall (pp. 19–37) on the general use of mathematical models and on the classification of different types of measurement (orderings, ratio scales, interval scales, etc.).

The next five papers deal with two classes of choice problems which have a similar structure: social choice, where the individual preferences are amalgamated to form a social preference scale (preferences being assumed represented by complete orderings), and choice among alternatives, where the consequences depend upon some unknown state of nature (the statistical decision problem). The two problems are formally analogous if individuals in the first interpretation are identified with states of nature in the second; however, the conditions for a satisfactory amalgamation need not be the same in the two cases. L. A. Goodman (pp. 39–48) points out the analogy and introduces a general amalgamation formula which includes all known types; the properties of this are not studied. In a remarkable paper, J. W. Milnor (pp. 49–59) lists a number of axioms, each of which embodies a desirable property of amalgamation in the statistical decision problem. He then shows how each of the various amalgamation formulas (minimax, minimax of regret, Laplacian, and others) can be characterized by a suitable subset of these axioms. This paper classifies all the different approaches to a theory of rational behavior under uncertainty in a uniquely revealing way, in the reviewer's opinion. Radner and Marschak (pp. 61–68) discuss a statistical example and apply two criteria, that of Hurwicz (a generalization of the minimax principle) and the minimax of regret principle; both lead to unsatisfactory results.

C. H. Coombs (pp. 69–86) studies social choice under the assumption that his "unfolding" technique is applicable for each individual [A theory of psychological scaling, Univ. of Michigan Press, 1952], i.e., the social alternatives can be regarded as forming a one-dimensional variable, x, and the utility of the ith individual is $-|x - x_i|$, where x_i is a parameter of the individual, the unit of utility being common to all. Two alternative methods, one based on equal voting weights for all and one on Thurstone's Law of Comparative Judgment, are briefly described, and the results compared in an experimental situation. Since the assumption of a common unit of utility removes the heart of the social choice problem, the purpose of the paper is somewhat obscure. S. (Valavanis-)Vail (pp. 87–98) makes a rather interesting classification of alternative interpretations of the concept of subjective probability; however, no applications are made, and the whole paper, like some others in this volume, seems very tentative.

The next three papers deal with learning theory as represented by recently developed mathematical models. R. R. Bush, F. Mosteller, and G. L. Thompson (pp. 99–126) formulate such a model; at each trial, there are a set of alternatives among which the subject must choose and a set of possible outcomes; the probability Π_{jk} that outcome k will follow the choice of alternative j is chosen by the experimenter and assumed constant among trials. Let $p(n)$ be the vector of probabilities of choice of different alternatives by the subject at the nth trial; as a result of choosing alternative j and observing outcome k, this vector is transformed into $p(n+1)=T_{jk}p(n)$, where the operator T_{jk} is assumed linear. The operator T_{jk} will be assumed to satisfy the "combining of classes" criterion, that if, for each k, the probabilities Π_{jk} are the same for all j in a certain class, then the class can be regarded as a single alternative for all purposes. Under these conditions, it is shown that each matrix T_{jk} can be expressed in the form $\alpha_{jk}I+(1-\alpha_{jk})\Lambda_{jk}$, where I is the unit matrix, all the columns of Λ_{jk} are the same, and $0\leqq\alpha_{jk}\leqq 1$. This representation is very useful in interpreting experiments; it also permits the development of a number of "trapping" theorems, stating regions in which the probability vector will ultimately lie.

W. K. Estes (pp. 127–137) considers a simple model which is equivalent to a special case of the preceding and applies it to the interpretation of an experiment. Individuals are asked to predict the occurrence or failure of an event whose occurrence is random with a constant probability Π; the theory states and the experiment confirms that individuals will tend after a number of trials to predict the event a proportion Π of the time and failure $1-\Pi$ of the time. "Rational" behavior would presumably call for predicting occurrence or failure, whichever is more probable, all the time; such a strategy yields a higher proportion of successful predictions. M. M. Flood has conducted some Monte Carlo experiments and also experiments on human subjects to test learning in game situations; the results are rather inconclusive.

G. Debreu (pp. 159–165) considers the representation of an ordering by a real-valued function, i.e., if X is a space ordered (weakly) by a relation $\leqq$, does there exist a real-valued function $\phi(x)$ such that $x\leqq y$ if and only if $\phi(x)\leqq\phi(y)$ (such a function is called order-preserving)? It is shown that if X is separable and connected and if, for every x' in X, the sets $\{x|x\leqq x'\}$ and $\{x|x\geqq x'\}$ are closed, then there exists a continuous order-preserving function. It is possible to remove the connectedness condition by strengthening that of separability. This result, mathematically of considerable interest, clears up a major gap in the usual presentations of utility theory in economics.

In the next two papers, M. Hausner (pp. 167–180) and R. M. Thrall (pp. 181–186) consider the effect of modifying the von Neumann-Morgenstern postulates for choice among probability distributions (prospects) by dropping the Archimedean postulate. It is then shown that, instead of a single utility function to represent the preferences, there will be a sequence of utility functions, the choice between two prospects being made in accordance with the first utility function in the sequence which gives different values to the two prospects. It is remarked that the theories of zero-sum two-person games and of linear programming are not essentially altered by the introduction of multi-dimensional utilities.

J. Marschak (pp. 187–220) applies the utility theory of choice under uncertainty to the problem of organization. It is assumed that there are a number of individuals, all of whom have the same utility function; these form a team. Each individual can make observations, take actions, and send messages to other members of the team, each message being a statement that the observations lie in a certain subset. The actions taken by an individual must depend upon the information available to him, that is, the observations he has made and the messages he has received. The messages are costly; the utility of the team then depends upon the actions taken, the state of the external world, and the messages sent. The problem is choose rules of action and communication to maximize expected utility. A number of examples are given to illustrate the type of problems encountered. The method of approach is very interesting; however, even very simple models give rise to complex mathematical problems.

H. G. Bohnert (pp. 221–230) objects to the logical structure of utility theory and suggests a reformulation making use of modal logic. While some of his objections to the empirical meaning of utility are sound, the reviewer does not understand what his positive contributions imply.

The last four papers are devoted to various experimental studies of decision. P. J. Hoffman, L. Festinger, and D. H. Lawrence (pp. 231–253) make an analysis of the outcome of an actual n-person game; they point out that psychological interactions among the players modify the game pay-offs, the outcomes being evaluated by comparison with others rather than absolutely. C. H. Coombs and D. Beardslee (pp. 255–285) describe in a general way the possible outcomes of experiments on decision-making under uncertainty and summarize a few of them. M. M. Flood (pp. 287–299) describes an experiment to interpret the results of Estes's work summarized above; he argues that the behavior of the subjects could be regarded as rational if they were not convinced that the probability of occurrence was constant over time. As Flood observes, the results of his experiment were inconclusive. In the last paper, G. Kalisch, J. W. Milnor, J. F. Nash, and E. D. Nering (pp. 301–327) describe some experimental n-person games and their agreement with some theories. The detailed descriptions are very interesting, but, as is to be expected, no broad generalizations emerge. *K. J. Arrow.*

*Balderston, J. **Models of general economic equilibrium.** Economic activity analysis, pp. 3–41. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

Brief expository account of the general equilibrium models of Walras, Leontief, von Neumann, and Wood-Dantzig. *R. Solow* (Cambridge, Mass.).

*Eckstein, O. **The input-output system—its nature and use.** Economic activity analysis, pp. 43–78. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

Expository paper. *R. Solow* (Cambridge, Mass.).

*Balderston, J. B., and Whitin, T. M. **Aggregation in the input-output model.** Economic activity analysis, pp. 79–128. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

Suppose in a Leontief input-output system, two industries, say i and j, are left unaggregated while the remaining industries are aggregated into some smaller number of groups. Sample computations are given which show that

the inverse element giving the direct and indirect effect of an increase in the final demand for commodity i on the total output of industry j may vary substantially, depending on how the remaining industries are aggregated. There is some discussion of practical and theoretical criteria for aggregation. *R. Solow* (Cambridge, Mass.).

***Morgenstern, Oskar. Experiment and large scale computation in economics.** Economic activity analysis, pp. 483–549. Edited by Oskar Morgenstern. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. $6.75.

***Ferreira Murteira, Bento José, and Amaro De Matos, António. On Giffen's paradox.** Trabalhos do seminário de econometria dirigido pelo Prof. H. O. Wold [Works of the seminar on econometrics led by Prof. H. O. Wold], pp. 167–185. Publicaçoes do Centro de Estudos Económicos, Lisbon, 1953. (Portuguese)

Consider a two-commodity indifference map defined by: $(y-f(x))x^{-\alpha}=\text{const.}$, where $\alpha>1$, $f'<0$, $f''>0$. The authors find that throughout the region of the plane in which this family actually represents a system of downward-sloping convex indifference curves Giffen's paradox occurs, i.e. $\partial x/\partial p_x>0$. *R. Solow* (Cambridge, Mass.).

TOPOLOGY

***Lietzmann, W. Anschauliche Topologie.** Verlag von R. Oldenbourg, München, 1955. 172 pp. DM 14.80.

The author's aim is to give an intuitive account of the concepts and problems of topology in terms of concrete objects. Such a book, if it were based on a contemporary and purely scientific outlook could be quite useful. What the present author achieves seems rather to be a description of a limited portion of topology as it existed in an earlier age, in days when, for example, the distinction between "one-sidedness" and "non-orientability" was not clearly understood. Since neither homology nor homotopy is mentioned, most of the problems which occupy present-day topologists could not even be hinted at. *P. A. Smith.*

Kaluza, Th., Jr. Beweis einer Vermutung von Herrn H. Hopf über die Numerierung der Eckpunkte gewisser unendlicher Graphen. Arch. Math. 6, 157–158 (1955).

It is proved that if each vertex of a connected infinite graph is of finite degree then the vertices can be enumerated so that each one is joined by an edge to one of its successors in the enumeration. *W. T. Tutte* (Toronto, Ont.).

Nagata, Jun-iti. On uniform convergence on uniform space. J. Inst. Polytech. Osaka City Univ. Ser. A. 5, 53–62 (1954).

The author defines by axioms the concept of uniform convergence of directed sets on topological spaces. If R is a T_1-space and A is a directed system, then any mapping $\phi_\alpha(p)$ of $A\times R$ into R is called a uniform directed system. Uniform convergence of a system $\phi_\alpha(p)$ is defined by means of 4 axioms too complicated to reproduce here. If R is a space with uniform structure, then the author's axioms become obvious consequences of the usual notion of uniform convergence. Conversely, if the author's axioms are satisfied, a compatible uniform topology can be defined. The author also defines and discusses the concept of a uniform filter, and characterizes metric spaces by means of convergence on countable subsets of the directed system A. [See also earlier papers by the author, Osaka Math. J. 1, 166–181 (1949); J. Inst. Polytech. Osaka City Univ. Ser. A. 4, 35–41 (1953); MR 11, 185; 16, 275]. *E. Hewitt.*

Nagata, Jun-iti. On uniform topology of functional spaces. J. Inst. Polytech. Osaka City Univ. Ser. A. 5, 87–95 (1954).

Let R be a space with uniform structure, where $\{\mathfrak{U}_\alpha\}_{\alpha\in A}$ ($\mathfrak{U}_\alpha=\{U_\alpha(x)\}_{x\in R}$) is a uniform neighborhood system for R. For a bounded real-valued function f on R and $S\subset R$, let $[f(S)]$ denote the closed interval $[\inf_{x\in S} f(x), \sup_{x\in S} f(x)]$. Let $C_u(R)$ denote the set of all bounded uniformly continuous real-valued functions on R. For $f, g\in C_u(R)$, $x\in R$, and $\alpha\in A$, let $d_\alpha(g(x), f(x))$ be the number

$$\max\{d(g(x), [f(U_\alpha(x))]), d(f(x), [g(U_\alpha(x))])\},$$

where $d(t, M)$ is the usual distance in the real number system between the point t and the set M. Now let $U_{\alpha\varepsilon}(f)$ ($\alpha\in A$, $\varepsilon>0$) be the set of all $g\in C_u(R)$ such that

$$d_\alpha(g(x), f(x))<\varepsilon$$

for all $x\in R$. These neighborhoods define a uniform structure on C_u, called the m-uniform topology. This topology is in general stronger than the weak topology and weaker than the usual uniform topology. The main theorem of the present paper asserts that 2 complete uniform spaces R_1 and R_2 are uniformly homeomorphic if and only if $C_u(R_1)$ and $C_u(R_2)$ are connected by a uniform homeomorphism that preserves order. A similar result holds for the space of uniformly continuous functions assuming values in the interval from 0 to 1. *E. Hewitt* (Seattle, Wash.).

Newns, W. F. Sur les espaces uniformes précompacts. Portugal. Math. 13, 33–34 (1954).

The author proves that the Bourbaki definition of precompact spaces is equivalent to the following one: a uniform space E is precompact if and only if it is a Hausdorff space in which every ultrafilter is a Cauchy filter.

J. Dieudonné (Evanston, Ill.).

Gillman, L., Henriksen, M., and Jerison, M. On a theorem of Gelfand and Kolmogoroff concerning maximal ideals in rings of continuous functions. Proc. Amer. Math. Soc. 5, 447–455 (1954).

The authors consider rings $C=C(X, R)$, $C^*=C^*(X, R)$ of continuous (bounded continuous) real-valued functions on a completely regular space X, as well as some properties of Hewitt's space νX [Trans. Amer. Math. Soc. 64, 45–99 (1948); MR 10, 126]. Main result (Hewitt's conjecture): every m-closed ideal in C is an intersection of maximal ideals (a set $M\subset C$ is m-closed if, for any $f\in C$, f non-ε M, there exists an everywhere positive $\pi\in C$ such that $g\in C$, $|g-f|\leq\pi$ implies g non-ε M); the corresponding theorem for C^* holds only for a pseudocompact X. It is proved that the relation $M^*\supset M\cap C^*$ defines a one-to-one correspondence between maximal ideals M^* in C^* and M in C (this simple result, essentially known, seems to be stated here explicitly for the first time). *M. Katětov* (Prague).

Shirota, Taira. On ideals in rings of continuous functions. Proc. Japan Acad. 30, 85–89 (1954).

A subring R of the ring of complex valued functions on a space X is called a normal function ring if it contains the

identity, the conjugate of each of its elements, the inverses of certain positive elements, and functions which separate certain sets in X. Let R_p be the set of all positive functions with inverses in R. The m-topology of R is defined by taking as neighborhoods of 0 sets of the form $\{f \in R \mid |f| < \pi\}$, $\pi \in R_p$. The main theorem is that an ideal of R is the intersection of maximal ideals containing it if (and only if) it is closed in the m-topology. This settles a problem of Hewitt [Trans. Amer. Math. Soc. 64, 45–99 (1948); MR 10, 126] for R the ring of all real-valued functions. Hewitt's conjecture was first verified by Gillman, Henriksen, and Jerison in the paper reviewed above, but the author's more general results were obtained independently. Both proofs depend upon the same basic idea, namely, that the ideal of all functions that vanish on a given set in X is the m-closure of the ideal of functions vanishing on a neighborhood of the set.

M. Jerison (Lafayette, Ind.).

Morales Martínez, Rodolfo. Demonstration of the equivalence of the topology of regular convergence and the k-topology for spaces of transformations. Bol. Soc. Mat. Mexicana 10, nos. 3–4, 23–28 (1953). (Spanish)

A proof of the reviewer's theorem (see title) [Ann. of Math. (2) 47, 480–495 (1946); MR 8, 165] is given, based on the uniform-structure analogue of the fact that in a metric space two disjoint closed sets of which one is compact are separated by a positive distance. *R. Arens.*

Corduneanu, C. Sur une théorème de Banach. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. 5, 39–43 (1954). (Romanian. Russian and French summaries)

Let S be a compact Hausdorff space, HS the group of all homeomorphisms of S onto S, BS the Banach space of all continuous real functions on S, and IBS the group of all isometries of BS onto BS which leave invariant the zero function and the unit function. For each $\varphi \in HS$, let $T_\varphi \in IBS$ be defined as follows: $T_\varphi f = f\varphi$ for each $f \in BS$. By a theorem of Eilenberg [Ann. of Math. (2) 43, 568–579 (1942); MR 4, 223], T maps HS onto IBS. The author observes that T is an anti-isomorphism, and is a homeomorphism if HS is taken in the topology of uniform convergence and IBS in the strong operator topology (i.e., the topology of pointwise convergence). (His discussion covers only the case of a compact metric S, but the theorem can be proved as just stated.) *V. L. Klee, Jr.* (Seattle, Wash.).

Martić, Ljubo. Quelques théorèmes sur l'ultrafiltre. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 9, 89–95 (1954). (Serbo-Croatian summary)

Proofs of well known properties of ultrafilters. Duality with maximal ideals of the boolean ring $\mathfrak{P}(E)$. Discussion and characterizations of the set $\mathfrak{P}(E) - \mathfrak{F}$, where $\mathfrak{F}$ is a filter on E. *P. Samuel* (Cambridge, Mass.).

Wagner, K. Zur Metrisierbarkeit topologischer Räume. Abh. Math. Sem. Univ. Hamburg 19, no. 1–2, 14–22 (1954).

The author proves a general numerical condition to be necessary and sufficient for the metrizability of a topological space. A number of well-known metrization theorems are shown to be easy consequences of this result. For the spaces considered, the topology is supposed determined by sequences of neighborhoods $U_n(p)$ having only the point p in common, for each point p. A valuation is a positive real-valued function $\epsilon_n(p)$ with the property that $\liminf \epsilon_n(p) = 0$ for each point p. A chain $K(p, q)$ joining two points is a finite sequence of neighborhoods $U_{n_1}(p_1), \cdots, U_{n_r}(p_r)$ whose consecutive pairs intersect, while $p_1 = p$ and $p_r = q$. The length of the chain is $\sum \epsilon_{n_k}(p_k)$, this notion being relative to a particular valuation.

The author's metrization condition is that a neighborhood space is metrizable if and only if there exists a valuation for which it is true that for each neighborhood $U_n(p)$, the greatest lower bound of the lengths of all chains leading from p to points q which are not in the neighborhood $U_n(p)$, is positive.

He shows that this result leads simply to the metrization theorems of Chittenden, Alexandroff and Urysohn, Niemytski, A. H. Frink, N. Aronszajn, and R. H. Bing. He does not discuss the recent metrization theorems of Smirnov [Dokl. Akad. Nauk SSSR (N.S.) 77, 197–200 (1951); Uspehi Mat. Nauk (N.S.) 6, no. 6(46), 100–111 (1951); MR 12, 845; 14, 70]. He also defines weak topological product of spaces, and uses his metrization theorem to show that the weak product of spaces, infinitely many of which have at least one non-isolated point, is not metrizable.

O. Frink (University Park, Pa.).

***Novák, Josef. Über die bikompakte Hülle einer isolierten abzählbaren Menge.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 280–283. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Let N be a countably infinite discrete space, and βN the Stone-Čech compactification of N. By examining $\beta N \cap N'$, the author establishes the existence of two countably compact Hausdorff spaces whose Cartesian product is not countably compact and of a non-compact completely regular space on which every continuous real-valued function is bounded. Using also the continuum hypothesis, he answers four questions posed by Luzin [Izv. Akad. Nauk SSSR. Ser. Mat. 11, 403–410 (1947); MR 9, 82]. *E. Hewitt.*

Inagaki, Takeshi. Contribution à la topologie. III. Math. J. Okayama Univ. 4, 79–96 (1954).

This paper contains §7, Borel sets, and §8, (B) measurable functions, of the author's treatise on those aspects of topology in which countability can be replaced by conditions involving other cardinal numbers. In the preceding part [same J. 2, 149–184 (1953); MR 14, 1001] he defined the character $\aleph_t$ of a uniform space X. Borel sets in X are here defined from closed and open sets by unions and intersections of families of $\aleph_t$ subsets of X. Montgomery's results on Borel sets in non-separable metric spaces and the corresponding properties of (B) measurable functions are derived, with considerable completeness, in this more general situation. *M. M. Day* (Urbana, Ill.).

Iséki, Kiyoshi. On a property of mappings of metric spaces. Proc. Japan Acad. 30, 570–571 (1954).

O. Hanner has shown [Ark. Mat. 2, 315–360 (1952); MR 14, 396] that every metric ANR (paracompact) is an absolute G_δ. The author observes that Hanner's method yields this result with "paracompact" replaced by "completely normal". [Reviewer's note: Hanner's method even yields this result with "paracompact" replaced by "completely paracompact" (i.e. every subset is paracompact).]

E. Michael (Seattle, Wash.).

Ford, Lester R., Jr. Homeomorphism groups and coset spaces. Trans. Amer. Math. Soc. **77**, 490–497 (1954).

For X compact, the topology of uniform convergence makes the group $H(X)$ of homeomorphisms into a topologic group. For X locally compact, the reviewer has proposed compactifying X by adding one point, and treating $H(X)$ as a subgroup of $H(X^*)$ [Amer. J. Math. **68**, 593–610 (1946); MR **8**, 479]. The author considers spaces x with a uniform structure, and proves that a subgroup S of $H(X)$ whose elements are all uniformly continuous, is topologic relative to uniform convergence. [One way of applying this to $G(X)$, X locally compact, amounts to regarding $G(X)$ as a subgroup of $H(X^*)$, where X^* is the Stone-Čech compactification.] The second topic deals with "reasonable" topologies for $H(X)$, which are those that make it topologic and also insure that the space of left cosets modulo the subgroup of stability of any point (and $H(X)$ is supposed transitive) is homeomorphic to X. Several necessary, and one sufficient condition are given. The latter is this: the space has a basis including the void set $\{U\}$ such that the subgroup H_U of $H(X)$ of those h in $H(X)$ that are the identity outside U is transitive on U. Then $H(x)$, if topologized as above, is "reasonable." Manifolds and normed linear spaces have this property, but it is not necessary for reasonableness. An example provided has the property of being arcwise connected, locally arcwise connected, and one-dimensional, but removal of any point gives three components. *R. Arens* (Los Angeles, Calif.).

Cohen, Haskell. A cohomological definition of dimension for locally compact Hausdorff spaces. Duke Math. J. **21**, 209–224 (1954).

The "codimension" $\operatorname{cd}(X, X_0)$ of a locally compact Hausdorff space X relatively to a closed subset X_0 is defined in the following way: (1) $\operatorname{cd}(X, X_0) = -1$ if X is void; (2) if X is compact non-void, then $\operatorname{cd}(X, X_0)$ is the inf of n such that, for any closed $C \subset X$, every $e \in H^n(C, C \cap X_0)$ admits of an extension to $H^n(X, X_0)$; H^n denotes the nth cohomology group (with coefficients from a given group G); (3) $\operatorname{cd}(X, X_0)$ is the sup of $\operatorname{cd}(K, K \cap X_0)$ for compact sets $K \subset X$. The author states that it is not known, except for special cases, e.g. polyhedra, whether $\operatorname{cd}(X, X_0)$ is independent of G.

If X_0 is void, put $\operatorname{cd} X = \operatorname{cd}(X, X_0)$. The following results make the meaning of $\operatorname{cd}(X, X_0)$ more clear:

$$\operatorname{cd}(X, X_0) = \operatorname{cd} Y,$$

where Y is obtained from X by identification of points of X_0; $\operatorname{cd} X \leqq \operatorname{cov} X$, the covering dimension of X, and $\operatorname{cd} X = \operatorname{cov} X$ if X is compact, G is the group of integers [cf. P. Alexandroff, Proc. Roy. Soc. London Ser. A. **189**, 11–39 (1947)].

The codimension is shown to possess many useful properties of $\operatorname{cov} X$ such as the sum theorem: if $X = \bigcup K_\lambda$, K_λ closed, $\operatorname{cd} K_\lambda \leqq n$, and either $\{K_\lambda\}$ is countable or every K_λ has arbitrarily small open neighborhoods U with $\operatorname{cd}(\bar{U} - U) \leqq n-1$, then $\operatorname{cd} X \leqq n$ (this result implies, of course, that $\operatorname{cd} X \leqq \operatorname{ind} X$, the inductive dimension of X). The author proves further that $\operatorname{cd}(X \times Y) \leqq \operatorname{cd} X + \operatorname{ind} Y$ if X, Y are compact non-void, and states that it is not known whether this inequality holds with $\operatorname{cd} Y$ instead of $\operatorname{ind} Y$; if $\operatorname{cd} X \geqq 0$, $\operatorname{cd} Y = 1$ and either $\operatorname{cd} X = \operatorname{ind} X$ or $\operatorname{cd} Y = \operatorname{ind} Y$, then $\operatorname{cd}(X \times Y) = \operatorname{cd} X + \operatorname{cd} Y$.

M. Katětov (Prague).

Lokucievskiĭ, O. V. Example of an open mapping of a one-dimensional compactum onto the Hilbert parallelepiped. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 118–130 (1954). (Russian)

It is proved that there exists an open map of a one-dimensional compact metric space onto the Hilbert parallelotope. No reference is made to the work of Anderson, who proved [Duke Math. J. **19**, 359–366 (1952); MR **14**, 71] the existence of such a map which is, in addition, monotone.

E. E. Floyd (Charlottesville, Va.).

White, Paul A. Regular convergence. Bull. Amer. Math. Soc. **60**, 431–443 (1954).

This is the published version of an invited address delivered before the American Mathematical Society. It is therefore mainly expository. The motivating notion, that of r-regular convergence in a metric space M, is defined as follows: A sequence of sets A_i which converge to a set A is said to converge to A r-regularly if for each $\epsilon > 0$ there correspond an integer N and a positive number δ such that if $n > N$ and z^j, $j \leqq r$, is a cycle of A_n carried by a set of diameter $< \delta$, then $z^j \sim 0$ on a subset of A_n of diameter $< \epsilon$. The work of various authors who have contributed to the theory of r-regular convergence over the past 20 years is reviewed, and a corresponding bibliography presented.

To date, the main results of the theory have centered about the classical type of problem concerning what properties P of the sets A_i are carried over to the limit set A. Prominent among the properties P studied to date are those where (1) P comprises the characterizing properties of an elementary configuration such as an arc or a simple closed curve, (2) P is a more general manifold-like property, and (3) P is a local connectedness property in the homology sense. In addition to descriptions of these results, special sections of the address are devoted to three related topics which have been investigated: r-regular convergence space (=a subspace K of the space of all closed subsets of a compactum M, points of K being the closed lc^r subsets of M with convergence meaning r-regular convergence in terms of these subsets), r-monotone transformation (=continuous mapping such that the inverse of each point of the image space is acyclic in the sense of homology in all dimensions $\leqq r$), and r-regular transformation (=a continuous mapping T of A onto B such that if $b_i \to b$ in B, then the sets $T^{-1}(b_i)$ converge r-regularly to $T^{-1}(b)$ in A). A concluding section contains brief mention of scattered results concerning types of convergence which may be derived from other properties (such as r-dimensional semi-local connectedness and local connectedness in the sense of homotopy) in the same way that the notion of r-regular convergence is derived from the local connectedness property. *R. L. Wilder.*

Boland, J. Ch. An effective definition of a connected, locally connected and punctiform plane set. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 522–524 (1954).

Such a set has been described by R. L. Wilder [Trans. Amer. Math. Soc. **29**, 332–340 (1927)]. While he does not say so, Wilder's construction is effective. In the paper under review, a square plus its interior is effectively decomposed into two sets, each of which intersects every nondegenerate continuum lying in their union. It follows that each of them has the prescribed properties. *F. B. Jones.*

de Groot, J. Example of two sets neither of which contains a continuous image of the other. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 525–526 (1954).

Let X be a connected, locally connected punctiform set and let Y be a nondegenerate, hereditarily indecomposable continuum. Such sets, X and Y, may be defined effectively in the plane. Neither X nor Y may be mapped non-trivially and continuously into the other. The author also gives another such situation where only one of the sets has been defined effectively. This is the case when X is the Knaster-Kuratowski example of a connected subset of the plane which contains an "explosion" point and Y is a connected, locally connected subset of the plane which contains no perfect set. *F. B. Jones* (Chapel Hill, N. C.).

Floyd, E. E. Orbit spaces of finite transformation groups. II. Duke Math. J. **22**, 33–38 (1955).

Let X be a finite-dimensional compact metric space and G a finite group of transformations on X. Let Y be the space of orbits of (G, X). Continuing an earlier work [Duke Math. J. **20**, 563–567 (1953); MR **15**, 456], the author shows that if X is homologically trivial relative to a compact coefficient group K, so is Y; if X is an absolute neighborhood retract, so is Y; if X is an absolute retract, so is Y. These results furnish an answer to a problem proposed by Borsuk and Ulam [Bull. Amer. Math. Soc. **37**, 875–882 (1931)] concerning symmetric products. *P. A. Smith.*

***Tornehave, H. Recent investigations on almost periodic movements.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 302–309 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Désignant par $[x, y]$ la distance de deux points d'un espace métrique complet M, celui-ci est dite uniformément localement convexe, s'il existe un nombre $\Delta>0$, tel que pour $[x, y]\leqq\Delta$, on puisse trouver une fonction: $z=\phi(x, t, y)$, continue en t $(0\leqq t\leqq 1; z \in M)$ telle que: $\phi(x, 0, y)=x$; $\phi(x, 1, y)=y$; $\phi(x, t, x)=x$. Dans un tel espace tout mouvement presque-périodique: $x=f(t)$ $(-\infty<t<+\infty; x \in M)$ peut être approché d'aussi près qu'on le veut par un mouvement: $x=g(\beta_1 t, \cdots, \beta_m t)$, où $x=g(u_1, \cdots, u_m)$ désigne un mouvement périodique de période 2 par rapport à chacune des variables u, et où les β sont rationnellement indépendants. Deux mouvements tels que $[f_1(t), f_2(t)]\leqq\Delta$ sont homotopes; un mouvement est dit réductible s'il est homotope à un mouvement périodique. Enoncé d'un résultat relatif aux mouvements réductibles. Exemples de mouvements non réductibles sur la sphére et le plan projectif. *J. Favard* (Paris).

Slye, John Marshall. Flat spaces for which the Jordan curve theorem holds true. Duke Math. J. **22**, 143–151 (1955).

Postulates for two spaces are given and the Jordan curve theorem is established in both spaces. In one space it is proved, and in the other space it is postulated, that when P and Q are two points, P and Q' are two points and R is a region containing P, then there exists a primitive simple closed curve in R that separates P and $Q+Q'$. The concept of flatness is contained in the definition of a primitive simple closed curve. If J is a simple closed curve, a component of the complement of J is said to be flat if it is true that when α and α' are two arcs lying, except for their endpoints, in the component and the endpoints of α separate those of α' on J, then α and α' intersect. A primitive simple closed curve is a simple closed curve J that separates the space into two connected sets each of which is a flat component of the complement of J.

Both spaces considered in the paper include the euclidean plane. However, there exist non-metrizable spaces and spaces lacking a nondegenerate separable continuum yet satisfying the postulates of the spaces considered. *W. R. Utz* (Columbia, Mo.).

Wille, R. J. Sur les espaces faiblement rétractiles. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 527–532 (1954).

The author is concerned with spaces S which have the property: given any (non-empty) closed $A\subset S$, there exists a mapping $f: S\to A$ which is the identity on A and is continuous (in S) at each point of A. It is proved that this property is implied by metrisability, implies complete normality, and is equivalent to neither, even for T_1 spaces with locally countable bases. [In Prop. 1, ρ_0 should be sup ρ, not inf ρ; in Example 1, the range of m should presumably be from 1 to 2^n.] *A. H. Stone* (Manchester).

Spanier, E. H., and Whitehead, J. H. C. On fibre spaces in which the fibre is contractible. Comment. Math. Helv. **29**, 1–8 (1955).

Let $f: X\to Y$ be a map and $A=f^{-1}(y_0)$ with $y_0 \in Y$. Assume that A is either a locally finite CW-complex or a compact metric ANR and that the covering homotopy theorem is valid with respect to f for any map $g: A\times A\to X$ and any homotopy of $f\circ g$. Then the main theorem of the paper states that, if A is contractible in X, it is an H-space, i.e. there exists a map $h: A\times A\to A$ such that $h(a, a_1)=h(a_1, a)=a$ for some a_1 and every a in A. Specializing to bundle spaces, it implies the theorem: If $f: X\to Y$ is a bundle map, if X is a locally compact, separable metric ANR and if a fibre A is contractible in X, then A is an H-space.

An interesting consequence of the main theorem is an application to the case that X is a cohomology n-sphere $(n\geqq 1)$ with integral coefficients. Then (1) either A is an AR or a cohomology q-sphere for some odd value of q with rational coefficients; and (2) if A is homeomorphic to a topological product, $A_1\times A_2$, then one of A_1, A_2 is an AR. Another by-product of the paper is the result: Any fibre bundle with a connected fibre, which, as a space, is a finite-dimensional, locally compact, separable metric AR, is equivalent to a product bundle. *S. T. Hu.*

Puppe, Dieter. Zur Homotopie von Abbildungen eines Polyeders. Math. Z. **61**, 303–323 (1954).

Let K denote a compact orientable pseudo-manifold of dimension n, σ_0 an n-simplex of K and $f_\alpha: K\to S^m$ a map such that $f_\alpha(K-\sigma_0)$ is constant and $f_\alpha|\sigma_0$ represents $\alpha \in \pi_n(S^m)$. The author considers the problem of determining when such a map f_α is essential. It is well known that, if $m=n$, f_α is essential if and only if $\alpha\neq 0$ and that, if m is even, $n=2m-1$ and K is a manifold, then f_α is essential if the Hopf invariant of α, $\gamma(\alpha)\neq 0$. The last result is extended to the case where K need not be a manifold (but only a pseudo-manifold), and it is shown that if neither $m=n$ nor m is even and $n=2m-1$ there is a K such that f_α is inessential for all α, and if m is even and $n=2m-1$ there is K such that f_α is inessential if $\gamma(\alpha)=0$. The proofs use the technique of obstruction theory together with properties of the space of paths on the

m-sphere. If $n>m$ and the suspension homomorphism $\pi_{n-1}(S^{m-1})\to\pi_n(S^m)$ is an onto mapping, then a compact orientable n-dimensional pseudo-manifold is constructed which cannot be mapped essentially on S^m. *E. Spanier.*

Hilton, P. J. A note on the P-homomorphism in homotopy groups of spheres. Proc. Cambridge Philos. Soc. **51**, 230–233 (1955).

The author considers the homomorphism

$$P:\ \pi_r(S^n)\to\pi_{r+n-1}(S^n)$$

defined by $P\alpha=[\alpha,\iota]$, the Whitehead product of α with a generator ι of $\pi_n(S^n)$. Letting E denote the suspension homomorphism and η_n a generator of $\pi_{n+1}(S^n)$ $(n\geqq 2)$, the main results are: (1) if $\beta\in\pi_r(S^{4k-1})$ is such that $2\iota\circ\beta=0$, then $P(\eta_{4k-2}\circ\beta)=0$, and (2) if also $\eta_{2k}\circ E^{4k+2}\beta\neq 0$, then $PE\beta\neq 0$. Some applications of these results to the homomorphism E are given. The proofs involve consideration of the relations between homotopy groups of spheres and Stiefel manifolds. *E. Spanier* (Chicago, Ill.).

Borel, Armand. Nouvelle démonstration d'un théorème de P. A. Smith. Comment. Math. Helv. **29**, 27–39 (1955).

P. A. Smith proved [Ann. of Math. (2) **39**, 127–164 (1938)] that if X is a compact finite-dimensional space which has the homology groups of an n-sphere over the integers mod p, p prime, and if T is a periodic map of period p on X, then the fixed-point set F of T has the homology groups mod p of a k-sphere, $-1\leqq k\leqq n$. The reviewer proved [Trans. Amer. Math. Soc. **72**, 138–147 (1952); MR **13**, 673] that if p is odd then $n-k$ is even. The proofs were based on the special homology groups of Smith. In this paper, the author proves the same theorems using the techniques of spectral algebra. Let Y denote the orbit space of (X, T), and π: $X\to Y$ the natural map. Then π maps F homeomorphically onto a subset of Y, also called F. Now $X-F$, upon which the integers mod p operate without fixed points, is a principal fibre space with structure group the integers mod p, with base space $Y-F$ and projection π. The proofs use the generated spectral algebra, together with the exact sequence of cohomology with compact supports and the cohomology theory of cyclic groups.

E. E. Floyd (Charlottesville, Va.).

Wu, Wen-tsün. On squares in Grassmannian manifolds. Acta Sci. Sinica **2**, 91–115 (1953).

This paper gives detailed proofs of results announced in C. R. Acad. Sci. Paris **230**, 918–920 (1950); MR **12**, 42. Let W_i be the ith Stiefel-Whitney class mod 2, $i=1,\cdots,m$, in the Grassmann manifold $G_{n,m}$ (with the Stiefel manifold $V_{m+n,m}$ as principal bundle); Sq^i denotes the Steenrod square operations. The main result is

$$\mathrm{Sq}^r W^s=\sum_{t=0}^{r}\binom{s-r+t-1}{t}W^{r-t}\cdot W^{s+t}\quad(s\geqq r\geqq 0),$$

and the same formula for the "dual" classes $\overline{W}_j$, $j=1,\cdots,n$. The paper begins with a presentation and refinement of the results of Ehresmann [J. Math. Pures Appl. (9) **16**, 69–100 (1937)] and Chern [Ann. of Math. (2) **49**, 362–372 (1948); MR **9**, 456] on the cohomology mod 2 of $G_{n,m}$. It also gives a proof of the duality relations $h^*W^i=\sum_{i_1+i_2=i}W_1^{i_1}\otimes W_2^{i_2}$ corresponding to the natural map

$$h: G_{n_1,m_1}\times G_{n_2,m_2}\to G_{n_1+n_2,m_1+m_2};$$

they are proved by computing the intersection numbers of appropriate cycles. The proof of the main result is based on H. Cartan's formula [C. R. Acad. Sci. Paris **230**, 425–427 (1950); MR **12**, 42] for the Steenrod square in a product space. This formula has the effect that under the map h^*, mentioned above, the relations map into similar relations in G_{n_i,m_i}, where they hold by induction assumption; the relations hold then in $G_{n,m}$, because h^* has been shown before to be an isomorphism up to the appropriate dimension. [Another proof of Wu's result, utilizing universal bundles and spectral sequences, but formally along the same lines as Wu's proof, has been given by A. Borel, Comment. Math. Helv. **27**, 165–197 (1953); MR **15**, 244.] *H. Samelson.*

GEOMETRY

Beatty, S. Upper and lower estimates for the area of a triangle. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 1–5 (1954).

Gerretsen [Nieuw Tijdschr. Wiskunde **41**, 1–7 (1953); MR **15**, 148] proved that the area Δ of a triangle with sides a, b, c satisfies the inequalities $K-H\geqq(2\sqrt{3})\Delta\geqq 3K-5H$, where $H=\frac{1}{2}(a^2+b^2+c^2)$ and $K=bc+ca+ab$, with equality only when the triangle is equilateral. The author points out that the second inequality is significant only when $3K>5H$, in which case a stronger inequality in the same direction is $12\Delta^2\geqq(K-H)(3K-5H)$. For instance, in the case of the famous triangle with sides 3, 4, 5 and area 6, Gerretsen's inequality gives $\Delta^2>64/3$ while the new inequality gives $\Delta^2>88/3$. *H. S. M. Coxeter* (Toronto, Ont.).

Cavallaro, Vincenzo G. Formules remarquables pour le triangle des centres des carrés construits sur les côtés d'un triangle. Mathesis **63**, 357–363 (1954).

Blanchard, René. Sur les points de Feuerbach. Mathesis **63**, 349–356 (1954).

Goormaghtigh, R. Sur l'hexagone inscriptible. Mathesis **63**, 335–338 (1954).

Rényi, Alfred. Remark on the angles of a polygon. Časopis Pěst. Mat. **78**, 305–306 (1953). (Czech)

The angles $\alpha_1,\cdots,\alpha_n$ of a simple closed polygon in E^2 satisfy $0<\alpha_i<2\pi$ and $\sum\alpha_i=(n-2)\pi$. If numbers α_i with these properties are given, then a polygon with the α_i as angles in the given order exist. *H. Busemann* (Paris).

Marmion, A. Sur les sphères podaires par rapport à un tétraèdre. Mathesis **63**, 339–349 (1954).

Bottema, O. A generalization of Pascal's theorem. Duke Math. J. **22**, 123–127 (1955).

The author proves the following converse of a theorem due to Kollros [Comment. Math. Helv. **19**, 316–319 (1946); MR **8**, 484]: Let two simplexes $A_1\cdots A_{n+1}$ and $W_1\cdots W_{n+1}$, formed by hyperplanes $\alpha_1\cdots\alpha_{n+1}$ and $w_1\cdots w_{n+1}$, be so situated that the $n+1$ $(n-2)$-spaces $\alpha_i\cdot w_i$ are associated (i.e., every line that meets n of them meets the remaining one also); then the $n(n+1)$ points $A_iA_k\cdot w_i$ $(i\neq k)$ lie on a quadric. He gives an analytic proof from which it appears (though he does not say so) that $A_1\cdots A_{n+1}$ and $W_1\cdots W_{n+1}$ are polar simplexes for a certain symmetric polarity. He has thus incidentally proved the converse of a theorem on polar simplexes due to Baker [Proc. Cambridge Philos. Soc. **32**,

507–520 (1936), p. 510; cf. S. Beatty, Amer. Math. Monthly 51, 599 (1944)]. The author remarks that the three-dimensional case of Kollros's theorem was stated by Chasles [Aperçu historique sur l'origine et le développement des méthodes en géométrie, 3me éd., Gauthier-Villars, Paris, 1889, p. 400], proved by Salmon [Analytic geometry of three dimensions, vol. 1, 6th ed., Longmans-Green, London, 1914, p. 142], and rediscovered by Court [Duke Math. J. 20, 417–421 (1953); MR 15, 245]. *H. S. M. Coxeter.*

De Donder, Th. Les trigonométries non euclidiennes déduites de la définition générale de l'angle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 41, 8–11 (1955).

The author verifies that his general definition of angle [same Bull. (5) 18, 303–310 (1932), p. 307] yields a familiar trigonometrical formula when applied to a two-dimensional space of constant curvature. In the spherical case he uses geodesic polar coordinates, so that $ds^2=(dx^1)^2+B(dx^2)^2$, where $B=\sin^2 x^1$. It is surprising that he does not use the analogous coordinates with $B=(x^1)^2$ in the Euclidean case and $B=\sinh^2 x^1$ in the hyperbolic case.

H. S. M. Coxeter (Toronto, Ont.).

***Bompiani, E. On Poincaré's representation of the hyperbolic space on an Euclidean half-space.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 15–19. Academic Press Inc., New York, 1954. $9.00.

Poincaré's representation of the proper points of the hyperbolic space H on those of an Euclidean half-space $\mathcal{S}$ may be written as follows: $X_0=x_0+x_3$, $X_1=x_1$, $X_2=x_2$, $X_3=\sqrt{\Omega(x)}$. Here x_i are the projective coordinates in H, $\Omega(x)\equiv x_0^2-x_1^2-x_2^2-x_3^2=0$ is the equation of the absolute quadric, x_i/x_0 $(i=1, 2, 3)$ are Cartesian coordinates in $\mathcal{S}$, the half-space being $X_3>0$. The exceptional (proper) point $U(1, 0, 0, 0)$ in H corresponds to $(0, 0, 1)$ in $\mathcal{S}$. The points of the limiting plane $X_3=0$ correspond to those of the absolute and the author studies this correspondance more in detail. The main part of the paper considers in the plane at infinity $\mathfrak{U}$ of $\mathcal{S}$. For $X_0=0$ we have $x_0+x_3=0$ the tangent plane Ω in $S(1, 0, 0, -1)$; the corresponding element of S is therefore not a point but a plane. The author considers a curve k through S; if a point goes to S along k the corresponding point tends to $(0, 0, 0, 1)$ in $\mathcal{S}$ unless the tangent of k in S lies in the tangent plane of Ω. Therefore the (∞^2) points of $\mathfrak{U}$ correspond with the (∞^2) elements of the second order at S and tangent to Ω. Considering the hypercylinders having US as axis, the representation is made clear.

O. Bottema (Delft).

Ghosh, N. N. A matrix treatment of four-dimensional rotation in hyperspace. Proc. Nat. Inst. Sci. India 20, 542–547 (1954).

The author begins with the representation of a rotation in R_n by the matrix e^{Ω}, where Ω is a skew-symmetric matrix. For this representation he refers to his own paper [Bull. Calcutta Math. Soc. 40, 116–122 (1948); MR 10, 563]. In particular, he considers now the case of a four-dimensional rotation in R_n $(n\geq 4)$, which means that Ω has rank 4. By means of the identity $\Omega^5+p_1\Omega^3+p_2\Omega=0$ he reduces e^{Ω} into a polynomial of degree 4 in Ω, viz. $E+\sum_{r=1}^{4} f_r\Omega^r$, where the f_r are certain integral functions in p_1, p_2, which he expresses in terms of the non-zero roots $\pm i\beta_2$, $\pm i\beta_4$ of Ω. From here he derives some essentially trigonometrical identities between the f_r. Moreover, he arrives at an explicit formula for e^{Ω} that sets in evidence the well known fact that a four-dimensional rotation is composed of two plane rotations. *H. Schwerdtfeger* (Kingston, Ont.).

Kustaanheimo, P., and Qvist, B. Finite geometries and their application. Nordisk Mat. Tidskr. 2, 137–155, 191 (1954). (Swedish. English summary)

The authors define block-designs, thence finite geometries, and finally algebraic finite geometries with elements in a finite field. Since quadratic residues in a field in which -1 is a non-residue have all the standard properties of positive elements in an ordered field except that the sum of two residues is not necessarily a residue, it is possible, by choosing the finite field so that every integer up to some fixed large integer is a residue, to ensure that the usual properties of an ordered field fail only for large elements and their reciprocals. Some physical applications are suggested: the use of such a finite geometry for physics introduces a discreteness in some ways analogous to the quantum.

H. A. Thurston (Bristol).

Convex Domains, Extremal Problems, Integral Geometry

Hammer, Preston C. Maximal convex sets. Duke Math. J. 22, 103–106 (1955).

For a point x of a real linear space L, an "x-cone" is defined to be a convex cone in $L\backslash\{x\}$ which has vertex x, and a "semispace at x" is a maximal x-cone. For a semispace S at x, S^* denotes the set $2x-S$. The fundamental result, based on a theorem of J. W. Ellis [same J. 19, 417–421 (1952); MR 14, 146, 1277] is that if C is a convex set in $L\backslash\{x\}$, then C is contained in a semispace at x. This leads to a number of interesting corollaries, among them the following: (1) If K_1 and K_2 are disjoint x-cones, there is a semi-space S at x such that $S\supset K_1$ and $S^*\supset K_2$. (2) The semispaces at x form the minimal intersection basis for the family of all x-cones, and the semispaces in L form the minimal intersection basis for the family of all convex proper subsets of L. (3) If A and B are complementary convex sets in L, there is a semispace S at ϕ such that $A=\bigcap_{x\in B}(x+S)$ and $B=\bigcap_{y\in A}(y+S^*)$. (4) If C is a convex set in L, then $L\backslash C$ is convex if and only if the family of all translates of L is linearly ordered by inclusion. Also discussed is a relationship between semispaces and extreme points, and it is shown how the result (3) leads to a type of separation theorem valid for an arbitrary disjoint pair of convex sets. [A special case of (2) is given by T. S. Motzkin, mimeographed lecture notes on linear inequalities, U. C. L. A., 1951.] *V. L. Klee, Jr.* (Seattle, Wash.).

Bieri, H. Ein Minimum-Maximumproblem über konvexe Rotationskörper. Comment. Math. Helv. 28, 149–154 (1954).

The author considers convex surfaces of rotation and proves that for given polar distance l and given length L of the generating meridian curve, the minimum of the integral M of the mean curvature is attained only by the cylinder and the maximum by the double cone. Specifically, M satisfies the inequality

$$\pi^2(L-l)/2\leq M\leq \pi \arccos(l/L)\sqrt{(L^2-l^2)}.$$

By approximation and continuity considerations, it is permitted to restrict the attention to those surfaces of rotation

which are obtained by rotating polygons, provided one allows for the finite contribution which an edge contributes to the integral mean curvature. Such surfaces are made up of frustra of cones for which M is readily calculated. By the effect of coalescing adjacent frustra, the above inequalities are obtained. *J. W. Green.*

Bieri, H. Zwei Minimumprobleme über konvexe Rotationskörper. Elem. Math. 9, 63–67 (1954).

For a convex body of revolution, let l be the polar distance, L the length of the generating meridian curve, F the surface area, and V the volume. The author proves that for given l and L, the frustrum of a cone and only this gives the least value of F. The minimum value of V is attained for a cylinder, a proper frustrum of a cone, and a cone according as $0 \leqq l \leqq 4l/3$, $4l/3 < L < \sqrt{2}l$, or $\sqrt{2}l \leqq L < \infty$. By considerations of approximation and continuity, it is permissible to consider meridian sections which are polygons. The proofs are made by observing the effect of coalescing adjacent cone frustra into a single one. *J. W. Green.*

Hadwiger, H. Ueber gemischte Quermassintegrale konvexer Rotationskörper. Rev. Fac. Sci. Univ. Istanbul. Ser. A. 19, 98–105 (1954). (Turkish summary)

Let A and B be convex bodies in euclidean κ-space; K the unit sphere. Let $W_{\nu\mu}(A, B)$ denote the mixed volumes

$$W_{\nu\mu}(A, B) = V(\underbrace{A, \cdots, A}_{\kappa-\nu-\mu}, \underbrace{B, \cdots, B}_{\mu}, \underbrace{K, \cdots, K}_{\nu})$$

[$=\nu$th mixed "Quermass" integrals of A and B]. Let $0 \leqq \alpha < \beta < \gamma \leqq \kappa - \nu$. Fenchel's inequality $W^2_{\nu\mu} \geqq W_{\nu,\mu-1} W_{\nu,\mu+1}$ readily implies (1) $W^{\beta-\gamma}_{\nu\alpha} W^{\gamma-\alpha}_{\nu\beta} W^{\alpha-\beta}_{\nu\gamma} \geqq 1$ [C. R. Acad. Sci. Paris 203, 647–650 (1936)]. The author considers the case that A and B are bodies of revolution whose common axis passes through the origin. Let a [b] denote the equatorial radius of A [B]. Then the following improvement of (1) is obtained

$$(2)\quad (\beta-\gamma)(a/b)^{\alpha} W_{\nu\alpha} + (\gamma-\alpha)(a/b)^{\beta} W_{\nu\beta} + (\alpha-\beta)(a/b)^{\gamma} W_{\nu\gamma} \geqq 0.$$

The pairs A, B are characterized for which equality holds in (1) or (2). *P. Scherk* (Saskatoon, Sask.).

Danelevič, I. A. Unique determination of unbounded convex polyhedra in a Lobačevskiĭ space. Mat. Sb. N.S. 35(77), 569–573 (1954). (Russian)

An unbounded convex polyhedron P (with a finite number of bounded or unbounded faces) in a 3-dimensional hyperbolic space H^3 is said to have a finite number of points at infinity, if each point of P is origin of at most a finite number of rays lying entirely on P. If two such polyhedra in the same H^3 are intrinsically isometric, then they are congruent as sets in H^3. Sufficient conditions are given for the same assertion to hold for convex polyhedra with an infinite number of points at infinity. The proofs are based on a generalization of Cauchy's lemma. For bounded polyhedra the assertion is found in A. D. Alexandrov's book [Convex polyhedra, Gostehizdat, Moscow-Leningrad, 1950; MR 12, 732]. *H. Busemann* (Paris).

Laugwitz, Detlef. Konvexe Mittelpunktsbereiche und normierte Räume. Math. Z. 61, 235–244 (1954).

The author obtains properties of the indicatrix in a normed space and gives a characterization of euclidean space. Let 2Π be the circumference of the indicatrix (set of vectors x for which $\|x\| = 1$) in a two-dimensional normed space, that is, in a Minkowski plane. Then it is proved that $3 \leqq \Pi \leqq 4$ with all possibilities realized for various spaces. Necessary to the proof is the theorem that every convex curve with midpoint possesses at least two pairs of conjugate diameters. Suppose now that the indicatrix has a second derivative $x''(s)$ with respect to Minkowski arc length s. Then for each s one has $x''(s) = -\kappa_r(s)x(s) + \kappa_l(s)x'(s)$. A Radon curve is one which possesses a conjugate diameter to every diameter. The author proves that the indicatrix is a Radon curve if and only if $\kappa_l(s) = \text{const.}$ The euclidean planes are characterized by the fact that both $\kappa_l(s) = \text{const.}$ and $\kappa_r(s) = \text{const.}$ It is shown that the contour integral $\oint \|dx'(s)\|$ of a closed curve $x(s)$ is $\geqq 2\Pi$ and $= 2\Pi$ precisely for convex curves. Also studied are "metric" spaces which are vector spaces with a star-domain determining the points $\|x\| \leqq 1$, $\|x\|$ being otherwise positively homogeneous, that is, $a > 0$ implies $\|ax\| = a\|x\|$. It is proved that a metric space is euclidean when every two-dimensional subspace is euclidean. *E. R. Lorch* (New York, N. Y.).

Pólya, G. An elementary analogue to the Gauss-Bonnet theorem. Amer. Math. Monthly 61, 601–603 (1954).

The author considers a polyhedral surface P of the topological type of a circular disk bounded by a closed polygon R. In the proof P is further restricted to be convex, although it is left open whether this is necessary; R may be skew. Denoting by T the sum of the changes of direction of R at its vertices and by S the area of the spherical image of P (defined in a natural way), the author proves simultaneously the Euler polyhedral formula $F - E + V = 1$ and the Gauss-Bonnet formula $S + T = 2\pi$. The proofs are simple and intuitive and rest on the formula for the sum of the angles of a plane polygon, together with a lemma equating the sum of the face angles of a polyhedral angle to 2π plus the measure of the solid angle included by its polar angle. *W. M. Boothby* (Evanston, Ill.).

Algebraic Geometry

Primrose, E. J. F. A property of quartic curves with two cusps and one node. Edinburgh Math. Notes no. 39, 1–3 (1954).

It is shown that a quartic curve with two cusps and one node is self-polar with respect to each of two conics. *G. B. Huff* (Athens, Ga.).

Granát, Luděk, and Fiedler, Miroslav. Rational curves with a maximum number of nodes. Časopis Pěst. Mat. 79, 157–161 (1954). (Czech)

Examples are given of rational curves of order n having the maximum number, $(n-1)(n-2)/2$, of nodes. One of them is $T_{n-1}(x) - T_n(y) = 0$, where T_n is the nth Tchebyshev polynomial. *F. A. Behrend* (Melbourne).

Godeaux, Lucien. Sur la surface cubique touchant un plan le long d'une droite. Mathesis 63, 326–327 (1954).

Godeaux, Lucien. Note sur les involutions cycliques appartenant à une surface algébrique. Arch. Math. 6, 1–4 (1954).

A brief restatement in four pages of the author's main results in the classifications of the united points of cyclic involutions, and of the corresponding isolated branch points

at singular points of a multiple surface, image of the involution. *P. Du Val* (London).

Ghezzo, Santuzza. Sulla dimensione della serie doppia di una data serie lineare. Rend. Sem. Mat. Univ. Padova **23**, 398–406 (1954).

Castelnuovo [Rend. Circ. Mat. Palermo **7**, 89–110 (1893) = Memorie scelte, Zanichelli, Bologna, 1937, pp. 95–113] showed that the minimum double of a linear series g_n^ρ on an irreducible algebraic curve, i.e. the series traced by quadrics on the projective model C^n in S_ρ of g_n^ρ, has dimension at least $3\rho-1$, and that its dimension is exactly $3\rho-1$ if and only if C^n lies on a surface of order $\rho-1$, i.e. a normal rational ruled surface or the Veronese surface. It is here proved that a necessary and sufficient condition for the dimension of the minimum double to be exactly $3\rho-1$ is that C^n has ∞^ρ n-secant normal rational $(\rho-1)$-ic curves, i.e. that almost all prime sections of C^n lie on such a curve, where by almost all is meant all except possibly a subsystem of lower dimensions. From this it follows that this condition is also sufficient (it is obviously necessary) for C^ρ to lie on a surface of order $\rho-1$. (The last result was proved by Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 374–378 (1952); MR **14**, 789] for $\rho=3$.)

P. Du Val (London).

Franchetta, Alfredo. Sulle forme algebriche di S_4 aventi l'hessiana indeterminata. Rend. Mat. e Appl. (5) **14**, 252–257 (1954).

Détermination des hypersurfaces F de l'espace projectif P_4 (sur un corps de caracteristique 0) dont la hessienne est indeterminée. Si F n'est pas un cône, elle contient une famille de plans a un paramètre, tous tangents à une courbe plane rationnelle. *P. Samuel* (Cambridge, Mass.).

Galafassi, Vittorio Emanuele. Sulle curve di diramazione dei piani tripli reali. Rend. Mat. e Appl. (5) **14**, 192–199 (1954).

Si une courbe de diramation C, l'est d'un plan triple, elle est d'ordre $6m$ et appartient à un faisceau dont les bases sont une courbe triple d'ordre $2m$ et une courbe double d'ordre $3m$; elle possède $6m^2$ cuspides, en général distincts; si C est à coefficients réels, il en est de même des courbes-base. Une telle courbe ne peut posséder que des branches paires ne se rencontrant pas, et un nombre pair de cuspides réels. Par variation infiniment petite à partir de types dégénérés, on peut montrer: Il existe toujours des courbes ayant le nombre maximum $(12m^2-9m+2)$ de branches dont $3m^2$ possèdent chacune 2 cuspides réels. Il existe des courbes C dont seules les $3m^2$ branches contenant des cuspides sont réelles. Il existe des courbes C n'ayant aucune branche réelle. La même méthode des petites variations permet d'obtenir d'autres types de ces courbes. *B. d'Orgeval* (Alger).

Spampinato, Nicolò. Varietà determinata da una coppia ordinata di ipersuperficie dell'S_r complesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **20** (1953), 58–65 (1954).

Dans un espace projectif S_{2r+1} à $2r+1$ dimensions, on a deux espaces indépendants à r dimensions, S_r' et S_r''; soit $x_1, y_1, \cdots, x_{r+1}, y_{r+1}$ les coordonnées d'un point de S_{2r+1}, les points fondamentaux de rang impair étant sur S_r' et ceux de rang pair sur S_r''. On considère alors les deux hypersurfaces V^n_{2r} et W^n_{2r} représentées par les équations suivantes:

$$f(x_1, \cdots, x_{r+1})=0; \quad g(x_1, \cdots, x_{r+1})+\sum y_i \frac{\partial f}{\partial x_i}=0;$$

où f, g sont deux formes de degré n. V^n_{2r} est un cône de sommet S_r'' projetant la variété V^n_{r-1} qui est donnée sur S_r' par l'équation $f=0$. W^n_{2r} passe par S_r'' avec la multiplicité $n-1$, et se compose de ∞^r espaces générateurs S_r, chacun rencontrant S_r'' en un espace S_{r-1}. L'A. se propose ici d'étudier la variété $V^{n^2}_{2r-1}$ intersection de V^n_{2r} et de W^n_{2r}; elle se compose toujours de ∞^{r-1} espaces S_r générateurs de W; dans le cas simple où la forme g est identiquement nulle, on obtient cette variété d'intersection en joignant tout point P' de V'^n_{r-1} avec l'espace S_{r-1} qui est tangent à la variété V''^n_{r-1} de S_r'', transformée de V'^n_{r-1} par l'homographie $\rho y_i=x_i$, au point P'' correspondant de P' dans la même homographie. L'A. ajoute encore d'autres propriétés sur les points singuliers de V'^n_{r-1}, sur le cas où les deux hypersurfaces $f=0$, $g=0$ coincident; et il donne enfin des équations paramétriques pour l'intersection VW. Ce couple de variétés V, W, avec leur intersection, se rencontre dans la représentation complexe d'une hypersurface algébrique de l'espace projectif S_r bidual; mais cette exposition est indépendante de la théorie des algèbres. *E. Togliatti* (Gênes).

Semple, J. G. The triangle as a geometric variable. Mathematika **1**, 80–88 (1954).

In a paper of Schubert's [Math. Ann. **17**, 153–212 (1880)], the enumerative calculus for triangles in a given plane is enabled to work only by the introduction of a mysterious classification of infinitesimal triangles. In this paper Schubert's approach is interpreted as a discussion of the base on a manifold representing triangles.

If a triangle is simply defined as a set of vertices A, B, C and edges a, b, c with the natural incidence relations, we obtain a variety W_6, representing the totality of triangles, which has a double threefold X representing all completely degenerate triangles (i.e. a triple vertex with an incident triple edge). To obtain a non-singular model two methods are suggested by the author. One, which is essentially Schubert's, is to consider not only A, B, C, a, b, c but also the net Σ of circumscribing conics. This gives rise to a non-singular W_6^*, birationally equivalent to W_6, and the transform X^* of X on W_6^*, is a fourfold. (In Schubert's approach a completely degenerate triangle is determined by the triple vertex A, the incident triple edge a and the radius of the circumcircle of the triangle, derived here from a three-point-contact net, thus making ∞^1 elements of X^* correspond to each element of X. Schubert's calculus is then interpreted simply as the theory of the base for subvarieties of W_6^*.)

The author suggests a second approach by taking a transform $\bar{W}_6$ of W_6 in which X is resolved not into a fourfold but transformed by a dilatation in the sense of B. Segre [Ann. Mat. Pura Appl. (4) **33**, 5–48 (1952); MR **14**, 683] into a fivefold variety. In this case a completely degenerate triangle requires for its description a further parameter which the author interprets as the limit of BC/AB as A, B, C tend to coincide. The author suggests the possibility of a triangle calculus different from Schubert's, derived from the base for subvarieties of W_6^*.

An attractive feature of the presentation of this paper is the brilliantly informal, and informative, manner in which is described the interrelations of the degeneration manifolds of W, W^* and $\bar{W}$. *D. B. Scott* (London).

Burniat, Pol. Sul genere lineare delle superficie algebriche. Rend. Mat. e Appl. (5) **14**, 23–29 (1954).

Soit $F_6=4F$ le plan quadruple abélien qui est défini par le champ de rationnalité

$$\{x, y, [\varphi_2(xy)\varphi_3(xy)]^{1/2}, [\varphi_1(xy)\varphi_3(xy)]^{1/2}\},$$

où x, y sont des coordonnées cartésiennes dans le plan F et les ϕ_i $(i=1, 2, 3)$ sont trois polynômes en x, y que l'on peut supposer de degré pair et sans composantes multiples. Soient E_i les courbes représentées par les équations $\phi_i=0$. Si e_r est une courbe moitié linéaire de E_i+E_j $(r\neq i\neq j)$, on peut exprimer le système canonique de F_0 à l'aide des courbes E_i et des systèmes adjoint $|e_r'|$. Après ces considérations générales, l'A. considère les deux cas particulier suivants: 1) E_1 est une courbe du 4ème ordre avec deux points doubles A et B; E_2 se compose de $2r+2$ droites issues de A; E_3 se compose de $2s+2$ droites issues de B; on suppose $r\geqq s\geqq 2$. On trouve alors, pour F_0, que: $p_g=p_a=rs$; $p^{(1)}=8p_g+1$. Pour toute valeur de p_g, qui ne soit pas un numbre premier; il existe donc des surfaces canoniques régulières qui ont $p^{(1)}=8p_g+1$. 2) E_1 se compose de deux droites se coupant en A; E_2 de $2r+2$ droites issues de A; E_3 de $2s+2$ droites issues d'un autre point B; on suppose encore $r\geqq s\geqq 2$. Dans ce cas on trouve pour F_0: $p_g=rs$, $p_a=p_g-r-1$, $p^{(1)}=8p_a+9$. Pour toute valeur de p_g qui ne soit pas un nombre premier, il existe donc des surfaces canoniques irrégulières qui ont $p^{(1)}=8p_a+9$.

E. Togliatti (Gênes).

Burniat, Poli. Surfaces algébriques de genre géométrique nul et de bigenre quelconque. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 459–463 (1954).

Soit F_0 une surface algébrique irréductible possédant une involution rationnelle I_6, d'ordre 4, composé avec trois involutions I_1, I_2, I_3, d'ordre 2, deux à deux permutables; F_0 équivaut à un plan quadruple $4F$ possédant trois courbes de diramation E_1, E_2, E_3 [voir l'analyse ci-dessus]. L'A. considère ici les trois cas particuliers suivants: 1) E_1 se compose de $2r+2$ droites d'un même faisceau $(r>0)$; E_2, E_3 sont deux coniques. Dans ce cas, on a pour F_0: $p_g=p_a=0$, $p^{(1)}=1$, $P_2=2r+1$. 2) E_1 se compose de $2r+1$ droites issues d'un même point O $(r\geqq 1)$; E_2, E_3 sont deux courbes du 3ème ordre possédant un même point double A et passant par O. On a alors pour F_0: $p_g=p_a=0$, $p^{(1)}=1$, $P_2=2r$. 3) E_1 se compose de $2r+4$ droites issues de O $(t\geqq 0)$; E_2, E_3 sont deux couples de droites issues d'un même point A. On trouve alors pour F_0: $p_g=0$, $p_a=-1$, $p^{(1)}=1$, $P_2=2r+1$; F_0 possède un faisceau elliptique de courbes de genre $2r+1$. Les deux premiers exemples prouvent que, pour toute valeur de P_2, il existe des surfaces régulières avec $p_g=0$, $p^{(1)}=1$. Le dernier exemple montre que, du moins pour toute valeur impaire de P_2, il existe des surfaces irrégulières pour lesquelles $p_g=0$, $p_a=-1$, $p^{(1)}=1$.

E. Togliatti (Gênes).

Burniat, Pol. Superficie algebriche ottuple canoniche di genere geometrico qualunque. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 326–331 (1954).

Soit H l'ensemble des fonctions rationnelles de x et de $\sqrt{f_1(x)}, \cdots, \sqrt{f_n(x)}$, où x est un point appartenant à une surface algébrique donnée F et les $f_i(x)$ sont des polynomes en les coordonnées du point x. On peut alors construire une nouvelle surface algébrique F_0 telle que les coordonnées d'un point de F_0 appartiennent à H. Sur F_0 il y a 2^n-1 involutions I_r d'ordre 2, que l'on obtient en changeant les signes des radicaux $\sqrt{f_i(x)}$; il y a aussi sur F_0 une involution J_0, d'ordre 2^n, composée avec chacune des I_r; J_0 est composée encore avec chacune 2^n-1 involutions J_s que l'on obtient en fixant le signe du produit d'un nombre quelconque des radicaux. Après quelques considérations générales sur la courbe de diramation de la surface 2^n-ple F (courbe qui s'obtient évidemment à partir des courbes de diramation des singles radicaux), et sur les systèmes canoniques et bicanoniques de F et de F_0, l'A. se place dans le cas $n=3$ et suppose que F soit un plan. Il considère ainsi quatre exemples différents de plans multiples d'ordre 8. Dans chacun d'eux, en partant de la courbe de diramation du plan multiple, il fait le calcul de p_a, p_g, $p^{(1)}$ et P_2. Les deux premiers exemples prouvent qu'il existent, pour toute valeur de $p_g>4$, des surfaces canoniques régulières multiples d'ordre 8. Les deux autres exemples conduisent à des surfaces d'irrégularité 1 et 3 respectivement et de genre géometrique pair; on en déduit l'existence, pour toute valeur paire de p_g, de surfaces canoniques irrégulières multiples d'ordre 8.

E. Togliatti (Gênes).

Filippi, Lidia. Su certe superficie generate da covarianti di forme binarie. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 285–289 (1954).

Let F be an algebraic surface of degree n and r a generic line of a general linear congruence with directrices s and t. Let $A_1, \cdots, A_n$ be the intersections of r and F. Then a set of points on r covariantly connected with $A_1, \cdots, A_n$ generates a surface G of degree ng; moreover, s and t are μ-tuple lines on G. Here g and μ are the resp. degree (grado) and weight (peso) of the covariant meant above. Furthermore, the following particular case is studied: F is a cubic which contains s and t and G is generated by the points which are determined by the Hessian of A_1, A_2, A_3.

F. J. Terpstra (Pretoria).

Guazzone, Stefano. Su certe sezioni spaziali di varietà intersezioni complete di due forme di S_r. Rend. Sem. Mat. Univ. Padova **21**, 293–302 (1952).

Ce mémoire s'apparente à un travail de Predonzan [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 238–242 (1948); MR **11**, 391], et à un mémoire antérieur de l'Auteur [Rend. Sem. Mat. Univ. Padova **21**, 243–251 (1952); MR **15**, 153]. L'Auteur démontre ici que la condition nécessaire et suffisante pour qu'une $V^{n(n-1)}$ intersection complète générique de deux formes F^n et F^{n-1} de S_r admette des sections par des S_k suivant des F_{k-1}^{n-1}, est que:

$$r+1\geqq k+\frac{1}{k+1}\binom{n+k}{k}.$$

La démonstration est conduite comme celles des deux mémoires cités, par un dénombrement de constantes avec vérification de la compatibilité et de l'indépendance des conditions dénombrées.

L. Gauthier (Nancy).

Morin, Ugo. Sull'unirazionalità dell'ipersuperficie del quarto ordine dell'S_6. Rend. Sem. Mat. Univ. Padova **21**, 406–409 (1952).

Il est connu par un exposé de B. Segre [Colloques Internat. Centre Nat. Rech. Sci., no. 24, 1950, pp. 83–91; MR **12**, 852], que l'hypersurface F^4_{r-1} de S_r est unirationnelle (c'est-à-dire admet une représentation paramétrique rationnelle, non nécessairement invertible) pour tout $r\geqq 7$.

L'Auteur démontre ici que ce résultat persiste pour $r=6$. Comme application d'un résultat de Predonzan [mêmes Rend. **21**, 335–344 (1952); MR **16**, 280], une F_5^4 de S_6 admet des S_3 qui la coupent suivant des F_2^4 monoïdales. Les tangentes inflexionnelles à F_5^4 aux points de F_2^4 recoupent chacune l'hypersurface en un seul point. L'Auteur, associant birationnellement les points de F_2^4, aux S_4 d'une étoile de sommet S_3^*, associe les tangentes inflexionnelles birationnellement aux points d'une V_5^m admettant S_3^* avec la multiplicité $m-2$. Cette dernière variété est rationnelle en vertu d'un résultat de Baldassarri [ibid. **19**, 1–43 (1950);

MR 13, 578], et la représentation unirationnelle de F_5^4 en résulte. La représentation n'est pas birationnelle: un point de F_5^4 admet 24 représentants dans S_5. La question de l'unirationnalité des F^4_{r-1} reste à examiner pour $3<r<6$.

L. Gauthier (Nancy).

Masotti Biggiogero, Giuseppina. Sulla composizione delle singolarità della hessiana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 354–364 (1951).

Après avoir étudié les singularités que présente la hessienne d'une courbe C en un point singulier O de celle-ci, à tangente unique (voir références ci-dessous), l'Auteur entreprend dans le présent mémoire l'examen du cas où C admet en O deux tangentes, dont l'une est l'origine d'une branche S présentant une des singularités déjà étudiées, et l'autre, l'origine d'une branche linéaire L sans inflexion.

Les trois cas envisagés pour S sont: S_1: le point O est r-uple, origine d'une branche superlinéaire d'ordre r et de classe 1. Le cas où l'espèce $s=1$ a été étudié par l'Auteur [Ann. Mat. Pura Appl. (4) **30**, 277–289 (1949); MR **12**, 735] avant celui de l'espèce s quelconque [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 444–458 (1950); MR **14**, 199]; S_2: le point O est double tacnodal d'espèce $s-1$; S_3: le point O est double de rebroussement d'espèce s et de classe 2. Ces deux cas ont été examinés par l'Auteur [ibid. **10(79)**, 89–96 (1946); MR **10**, 398].

Voici les résultats: Dans la superposition S_1+L, la hessienne présente deux branches d'ordre r et de classe 1 tangentes à S_1, une branche d'ordre $r-2$ et de classe 1 tangente à S_1, et une branche linéaire tangente à L. (La branche d'ordre $r-2$ disparaît pour $r=2$, et est linéaire pour $r=3$). Dans la superposition S_2+L, la hessienne possède en O trois branches linéaires distinctes, ayant un contact s-ponctuel (et pas plus) avec les branches du tacnode; une branche linéaire passant par les deux premiers points doubles du tacnode; et une branche linéaire tangente à L. Dans la superposition S_3+L, la hessienne possède un rebroussement de classe 2, d'espèce s ayant en commun avec S_3 le schéma $O^2, O_1^2, \cdots, O^2_{s-1}, O_s^1$; une branche linéaire passant par les $(s+1)$ points infiniment voisins cités; une branche linéaire passant seulement par les deux premiers, et une branche linéaire tangente à L.

Pour obtenir ces résultats l'Auteur applique les mêmes méthodes que dans ses mémoires antérieurs. L'équivalence plückérienne des singularités étudiées est donnée explicitement.

L. Gauthier (Nancy).

Du Val, Patrick. Transformations depending on sets of associated points. J. London Math. Soc. **30**, 24–32 (1955).

Room [Proc. Cambridge Philos. Soc. **48**, 383–391 (1952); MR **13**, 978] has considered a set of plane involutory Cremona transformations R_m which are in $(1;1)$ correspondence with the set of so-called focal curves m. In the present note the matrix notation of Room is replaced by the formalism of tensor-calculus in 10-space; for lowering and raising of suffixes the matrices Γ_{ij} and Γ^{ij} with $\Gamma_{00}=\Gamma^{00}=1$, $\Gamma_{ii}=\Gamma^{ii}=-1$ $(i\neq 0)$, $\Gamma_{ij}=\Gamma^{ij}=0$ $(i\neq j)$ are used. To each linear system of curves p there corresponds a vector p_i and to each R_m a tensor R_{ij} such that $q^i=p^jR_j{}^i$ corresponds to the image of p under R_m. Furthermore, $R_{ij}=m^kF_{kij}$, where m^k corresponds to the focal curve of R_m and where

$$F_{kij}=f_ku_iu_j+f_iu_ju_k+f_ju_ku_i-u_k\Gamma_{ij}-u_i\Gamma_{jk}-u_j\Gamma_{ki}.$$

Since f is the net of all lines and u a fixed pencil of cubics it follows that F_{kij} is the same for all R_m. The author defines G_{kij} which is found if in the latter expression f is replaced by any homaloidal net g. It is shown that the tensor $T_{ij}=m^kG_{kij}$ determines an involution of the set R_m. In the second part of the paper analogous formulas concerning Room's involutions in space are given.

F. J. Terpstra.

Barsotti, Iacopo. Il teorema di dualità per le varietà abeliane ed altri risultati. Rend. Mat. e Appl. (5) **14**, 98–114 (1954).

The author proves first that the Picard variety $P(A)$ of an abelian variety A is isogeneous to A, and that there exists a "purely inseparable" isomorphism of A onto $P(P(A))$ (this gives the duality theorem in characteristic 0). These results seem to be superseded by recent proofs of the duality between the Picard variety and the Albanese variety of any normal variety. Let now $W(A)$ denote the group of divisors X on A such that $t(X)\sim X$ for any translation t of A [cf. Weil, Variétés abeliennes, et courbes algébriques, Hermann, Paris, 1948, no. 57; MR **10**, 621]. It is proved that the group $G_a(A)$ of the divisors which are algebraically equivalent to 0 on A is a subgroup of $W(A)$, that there exists a power q of the characteristic exponent of the ground field such that $q\cdot W(A)=G_a(A)$, and that $G(A)/W(A)$ ($G(A)$ denoting the group of all divisors on A) is a free abelian group with a finite number of generators. In particular, the torsion σ of an abelian variety in characteristic 0 is equal to 1 (a result which was known by transcendental methods).

P. Samuel.

Shimura, Goro. Reduction of algebraic varieties with respect to a discrete valuation of the basic field. Amer. J. Math. **77**, 134–176 (1955).

This paper gives a timely account of the "reduction mod p" of algebraic varieties. The author begins by studying the specializations over a local domain $\mathfrak{o}$ (i.e. the specializations extending the natural homomorphism of $\mathfrak{o}$ onto its residue field $\mathfrak{o}/\mathfrak{m}$), their extensions and their specialization rings. Particular attention is given to the case where $\mathfrak{o}$ is a valuation ring. Multiplicities of such specializations are then studied, as in A. Weil's "Foundations of algebraic geometry" [Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; MR **9**, 303]. Let now k be a field, $\mathfrak{o}$ a discrete valuation ring of k, κ its residue field, V a variety (or a bunch) over k and $\mathfrak{a}$ its ideal. The image of $\mathfrak{o}[X]\cap\mathfrak{a}$ in $\kappa[X]$ is an ideal, and the corresponding bunch $\bar{V}$ is called the reduction of V. We have $\overline{U\cup V}=\bar{U}\cup\bar{V}$, $\overline{U\cap V}\subset\bar{U}\cap\bar{V}$, $\overline{U\times V}=\bar{U}\times\bar{V}$, and, if V is pure r-dimensional, then $\bar{V}$ is also pure r-dimensional.

Furthermore, let A be a rational cycle over k. By studying the intersection points of A with a generic linear variety of complementary dimension, and the multiplicities of their specializations over $\mathfrak{o}$, one defines a rational cycle $r(A)$ over κ, called the reduction of A. The mapping r is a homomorphism, whose behaviour with respect to products, projections and intersections is carefully studied, thus providing an extension of the theory of the specialization of cycles to the "unequal characteristic case". The preceding theory is carried over to the case of abstract varieties.

Finally, let $(\mathfrak{o}_\lambda)$ be a set of valuation rings of k such that every nonzero element of k is a unit in almost all $\mathfrak{o}_\lambda$. It is proved that an absolutely irreducible polynomial over k has an absolutely irreducible reduction over almost all the residue-fields $\mathfrak{o}_\lambda/\mathfrak{m}_\lambda$. It follows that, if V is any abstract variety over k, then the reduced cycle $r_\lambda(V)$ (relative to $\mathfrak{o}_\lambda$) is a variety for almost all λ.

P. Samuel.

Guggenheimer, Heinrich. Interpretazione topologica dei covarianti di Beniamino Segre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 331–334 (1954).

The object of this note is to show that covariant varieties $P_{s,i}$ of immersion of a variety P in the variety V are Steenrod squares. The sketch of the argument given is so condensed that the reviewer is unable to follow it in detail.

W. V. D. Hodge (Cambridge, England).

Guggenheimer, Heinrich. Omologia delle dilatazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 13–15 (1954).

If V is a non-singular algebraic variety of dimension v, P a non-singular sub-variety of it of domain p, V' a variety obtained from V by a birational transformation which transforms P into a variety P' of dimension $v-1$, then P' is a fibre variety whose fibres are projective spaces of S_r of dimension $r=v-p-1$. If $c_q(u)$ denotes the qth Chern class of the variety u, it is shown that

$$c_k(P')\cong\sum_{q=0}^{k}c_{k-q}(P)\times c_q(S_r).$$

W. V. D. Hodge (Cambridge, England).

Kodaira, Kunihiko. Some results in the transcendental theory of algebraic varieties. Ann. of Math. (2) **59**, 86–134 (1954).

Cet article a pour but d'établir des propriétés des variétés algébriques à l'aide de la théorie des intégrales harmoniques. Soit V une variété algébrique projective, de dimension n, sans singularité. Un système linéaire sur V est dit ample s'il n'a pas de composante fixe et si l'application de V dans un espace projectif qu'il définit est birégulière; pour $n\geqq 2$, en vertu des théorèmes de Bertini, le membre général d'un tel système est un ensemble analytique principal irréductible (diviseur premier) sans singularité. Notation: S (ou T) désignera un diviseur premier non singulier et $|D|$ le système linéaire complet défini par le diviseur D. Si K est le diviseur d'une forme différentielle méromorphe de degré n (diviseur canonique), $|K+D|$ est appelé le système adjoint de D. L'auteur démontre le théorème de Riemann-Roch pour les systèmes adjoints, dans les cas suivants: $D=S$ ($n\geqq 2$); $D=T+S$ ($n\geqq 3$ et le diviseur découpé par T sur S est premier non singulier); $D=E+S$ ($n\geqq 3$) où S et E sont des diviseurs premiers non singuliers, où E appartient à un système linéaire ample et découpe sur S un diviseur premier non singulier. Les énoncés obtenus font intervenir les sommes alternées

$$a(W)=g_m(W)-g_{m-1}(W)+\cdots+(-1)^{m-1}g_1(W),$$

où $g_i(W)$ désigne la dimension de l'espace vectoriel des i-formes différentielles de première espèce sur la variété W de dimension m et où W représente V ou l'un des diviseurs considérés. $\{E\}$ désignant le système linéaire ample des sections hyperplanes de V, pour h suffisamment grand, $|D_h|=|D+hE|$ est ample et $a(D_h)$ est un polynôme $a(h; D, V)$ en h, dont le terme constant $a_V(D)=a(0; D, V)$ ne dépend que de la classe d'équivalence linéaire de D. De plus, à l'aide du théorème de Riemann-Roch, on prouve que $\dim|D+hE|=v(h; D, V)=a(V)+a(h; D-K, V)-1$, donc $v(h; D, V)$ est un polynôme en h pour h assez grand. L'auteur montre l'égalité du genre arithmétique

$$P_a(V)=v(0; K, V)+1+(-1)^n$$

et de $a(V)$ [conjecture de Severi, Rend. Circ. Mat. Palermo **28**, 33–87 (1909)] et donne une propriété du genre arithmétique $p_a(V)=(-1)^n v(0; 0, V)$. Introduisant le résidu (de Severi) d'une fonction méromorphe uniforme ou additive (intégrale simple de seconde espèce) sur une composante S irréductible non singulière de son diviseur polaire, l'auteur construit une intégrale simple de seconde espèce dont le résidu est donné sur S et indique une condition nécessaire et suffisante sur les résidus pour que cette intégrale soit uniforme. Soit $|D|_S$ le système linéaire découpé par $|D|$ sur S et $|DS|_S$ le système linéaire complet qu'il définit sur S; l'expression $\text{def}\,(D/S)=\dim|DS|_S-\dim(|D|S)$ est appelée défaut de $|D|S$; alors, si $|S-D|$ est ample, le défaut de $|D|S$ est nul, d'où le lemme d'Enriques-Severi-Zariski [Zariski, Ann. of Math. (2) **55**, 552–592 (1952); MR **14**, 80] en prenant pour S, la section de V par une hypersurface générale d'ordre h. Appelons irrégularité de V la moitié q du premier nombre de Betti de V. Utilisant le théorème de Riemann-Roch pour les systèmes adjoints et des résultats récents sur les variétés de Picard et les fonctions thêta [A. Weil, Amer. J. Math. **74**, 865–894 (1952); MR **14**, 314], l'auteur montre que q est égal au maximum de $\text{def}\,(S/S)$ et qu'il existe exactement $2q$ intégrales simples de seconde espèce indépendantes sur V; puis, si $\mathfrak{C}$ est le système continu complet formé par les diviseurs effectifs homologues à un diviseur D_0 auquel on peut associer certains systèmes linéaires complets amples, l'auteur prouve que le système linéaire caractéristique de $\mathfrak{C}$ sur un diviseur premier non singulier S appartenant à $\mathfrak{C}$ est complet. Une dernière application du théorème de Riemann-Roch permet de montrer que tout revêtement fini non ramifié de V est une variété algébrique projective sans singularité (conjecture de A. Weil dans l'article cité). A titre auxiliaire, le théorème de Lefschetz [L'analysis situs et la géométrie algébrique, Gauthier-Villars, Paris, 1924] selon lequel tout k-cycle de V ($1\leqq k\leqq n-1$) à coefficients rationnels est homologue à un k-cycle de E, est démontré, au début de l'article, à l'aide des intégrales harmoniques; il en résulte, en particulier $g_k(E)=g_k(V)$ pour $1\leqq k\leqq n-2$.

P. Dolbeault (Paris).

Kodaira, K., and Spencer, D. C. On arithmetic genera of algebraic varieties. Proc. Nat. Acad. Sci. U. S. A. **39**, 641–649 (1953).

Soit $M=M_n$ une variété algébrique, non singulière, irréductible, de dimension n, immergée dans un espace projectif complexe et soit E la section hyperplane générale de M_n. Pour un diviseur arbitraire D de M_n, il existe un polynôme $v(h, D)$ en h, de degré n, tel que $\dim|D+hE|=v(h, D)$ pour h assez grand. Soit $|K|$ le système canonique, on appelle genre arithmétique de M_n, le nombre

$$P_a(M_n)=v(0; K)+1-(-1)^n.$$

Soit $p_a(M_n)=(-1)^n v(0; 0)$. Les auteurs prouvent, pour tout n, l'égalité $p_a(M_n)=P_a(M_n)$ qui n'était connue que dans des cas particuliers. Soit $g_s(M_n)$ la dimension de l'espace vectoriel des s-formes différentielles de première espèce sur M_n, on pose $a(M_n)=\sum_{s=0}^{n}(-1)^s g_s(M_n)$. Kodaira a prouvé: $P_a(M_n)=(-1)^n\{a(M_n)-1\}$ [Notes, Princeton University, 1953; et l'oeuvre analysée ci-dessus]. Soit D un diviseur arbitraire et D_h le membre général du système linéaire complet $|D+hE|$; pour h suffisamment grand, D_h est une sous-variété non singulière, donc $a(D_h)$ est bien défini. Il existe un polynôme $a(h; D, M_n)$ tel que $a(D_h)=a(h; D, M_n)$ pour h suffisamment grand. Soit $a_M(D)=a(0; D, M_n)$. Kodaira a montré (références précédentes) que $a_M(D)$ est indépendant de l'espace projectif ambiant et que (i) si $D=S$, sous-variété non singulière, on a: $a_M(S)=a(S)$; (ii) si $D\simeq D'$ ($\simeq$désigne l'équivalence linéaire), on a:

$a_M(D)=a_M(D')$; (iii) $a_M(D+S)=a_M(D)+a(S)-a_S(D\cdot S)$ où $D\cdot S$ désigne la sous-variété de S découpée par D; enfin, d'après le théorème de Riemann-Roch pour les systèmes adjoints, on a $p_a(M_n)=a(M_n)-a_M(-K)-(-1)^n$. Il réste donc à prouver: $a_M(-K)=[1-(-1)^n]a(M_n)$. Soit $\mathfrak{F}$ un faisceau analytique complexe sur une variété analytique complexe M de dimension n; si $\mathfrak{F}$ satisfait à la condition (F) $\{\dim H^s(M,\mathfrak{F})<+\infty$ pour $0\leqq s\leqq n$; $\dim H^s(M,\mathfrak{F})=0$ pour $s>n\}$, on définit la caractéristique d'Euler

$$\chi(M,\mathfrak{F})=\sum_{s=0}^{n}(-1)^s\dim H^s(M,\mathfrak{F}).$$

Soit $\Omega^r=\Omega^r(M)$ le faisceau des germes de r-formes holomorphes sur M, on pose $\chi^r(M)=\chi(M,\Omega^r)$. On sait que $H^s(M,\Omega^r)\approx H^{r,s}(M)$; donc si M est kählérienne compacte, la condition (F) est vérifiée pour $\mathfrak{F}=\Omega^r$, de plus

$$\dim H^{0,s}(M)=\dim H^{s,0}(M)=g_s(M)$$

et

$$\dim H^{n,s}(M)=\dim H^{n-s,0}(M)=g_{n-s}(M),$$

d'où: $\chi^0(M)=a(M)$; $\chi^n(M_n)=(-1)^n a(M_n)$. Soit D un diviseur de M et $\Omega_{D^r}=\Omega_{D^r}(M)$ le faisceau des germes de r-formes méromorphes multiples de $-D$; soit $\chi_{D^r}(M)=\chi(M,\Omega_{D^r})$. Il résulte de la Note de D. C. Spencer analysée ci-dessous que Ω_{S^n} satisfait à la condition (F) et que

$$a(S)=(-1)^{n-1}\{\chi_{S^n}(M_n)-\chi^n(M_n)\}.$$

Pour tout diviseur D, on pose

$$\bar{a}(D)=(-1)^{n-1}\{\chi_{D^n}(M)-\chi^n(M)\}. \tag{1}$$

De plus, si $D\simeq D'$ sur M, on a $H^s(M,\Omega_{D^n})\approx H^s(M,\Omega^n{}_{D'})$. Il s'agit de prouver d'abord que le second membre de (1) est défini, i.e. que Ω_{D^n} satisfait à la condition (F). On définit un homomorphisme $\mathfrak{P}$ tel que la suite

$$0\to\Omega_{D^n}\xrightarrow{i}\Omega^n{}_{D+S}\xrightarrow{\mathfrak{P}}\Omega^{n-1}_{D\cdot S}(S)\to 0$$

soit exacte et on fait la démonstration par récurrence sur n, à partir de la suite exacte de cohomologie associée. De plus

$$\chi^n{}_{D+S}(M)=\chi_{D^n}(M)+\chi^{n-1}_{D\cdot S}(S),$$

d'où $\bar{a}(D+S)=\bar{a}(D)+a(S)-\bar{a}(D\cdot S)$; enfin $\bar{a}(D)=a_M(D)$ (démonstration par récurrence sur n, en utilisant $a(T)=\bar{a}(T)$ où la variété non singulière T est le membre général de $|D+kE|$ pour k suffisamment grand). Si w est la n-forme différentielle méromorphe de diviseur $(w)=K$, l'application $f_p\to\tau_p=f_p\cdot w$, où $f_p\in(\Omega^0)_p$; $\tau_p\in(\Omega^n{}_{-K})_p$, induit l'isomorphisme $\Omega^0\approx\Omega^n{}_{-K}$, donc: $\chi^n{}_{-K}(M)=\chi^0(M)$, d'où

$$a_M(-K)=(-1)^{n-1}\{\chi^0(M)-\chi^n(M)\}=[1-(-1)^n]a(M),$$

c.q.f.d. On sait que $g_s(M_n)$ est un invariant birationnel, comme $p_a(M_n)=g_n(M_n)-g_{n-1}(M_n)+\cdots+(-1)^{n-1}g_1(M_n)$, il en est de même de $p_a(M_n)$. *P. Dolbeault* (Paris).

Spencer, D. C. Cohomology and the Riemann-Roch theorem. Proc. Nat. Acad. Sci. U. S. A. **39**, 660–669 (1953).

Considérons une variété analytique complexe M de dimension n et une sous-variété analytique non singulière S, de dimension $n-1$. Soient $\Omega^r(M)$ le faisceau des germes de r-formes holomorphes et $\Omega_{S^r}(M)$ le faisceau des germes de r-formes méromorphes multiples de $-S$. En tout point $p\in S$, soit $z_p{}^1,\cdots,z_p{}^n$ un système de coordonnées locales sur un voisinage U_p de p tel que $z_p{}^1=0$ soit l'équation locale de S et que $z_p{}^2,\cdots,z_p{}^n$ soient des coordonnées locales sur S, alors, sur $U_p\cap U_q\neq 0$, la fonction $z_p{}^1/z_q{}^1$ est holomorphe; désignons par f_{pq} sa restriction à S. On suppose qu'il existe un diviseur D sur S (diviseur caractéristique) et des fonctions méromorphes h_p sur $S\cap U_p$ telles que, si (h_p) désigne le diviseur de h_p, on ait $D\cap U_p=(h_p)$ et que $h_p/h_q=f_{pq}$; l'existence d'un diviseur caractéristique est assurée si M est une variété algébrique. Le germe de forme méromorphe $\tau_p\in\Omega_{S^r}(M)$ définit un germe de courant (généralisation de la valeur principale de Cauchy d'une fonction méromorphe); $\bar{\partial}$ étant la partie de type (0, 1) de l'opérateur de différentiation, on a $\bar{\partial}\tau_p=2\pi i\mathfrak{R}_p$, où $\mathfrak{R}_p$ est un germe de courant porté par S. L'ensemble des $\mathfrak{R}_p$ constitue un faisceau sur M désigné par $\mathfrak{R}_{S^r}$; on a l'isomorphisme canonique (1) $\mathfrak{R}_{S^r}\approx\Omega^{r-1}(S)+\Omega_{D^r}(S)$ et la suite exacte de faisceaux (2) $0\to\Omega^r\xrightarrow{i}\Omega_{S^r}\xrightarrow{\bar{\partial}}\mathfrak{R}_{S^r}\to 0$. La suite exacte de cohomologie définie par (2) fournit une expression de la dimension de l'espace vectoriel $H^s(M,\Omega_{S^r})$ en fonction des dimensions des espaces vectoriels $H^s(M,\mathfrak{R}_{S^r})$, $H^{s+1}(M,\Omega^r)$ et

$$H^t(M,\Omega^r)/\delta^*H^{t-1}(M,\mathfrak{R}_{S^r})$$

(où $t=s$, $s+1$ et où δ^* désigne l'homomorphisme de Bockstein). On sait que $H^s(M,\Omega^r)\approx H^{r,s}(M)$ et d'après (1), on a $H^s(M,\mathfrak{R}_{S^r})\approx H^s(S,\Omega^{r-1})+H^s(S,\Omega_{D^r})$. Soient

$$a^r(M)=\sum_{s=0}^{n}(-1)^s\dim H^s(M,\Omega^r)$$

et $a_{S^r}(M)=\sum_{s=0}^{n}(-1)^s\dim H^s(M,\Omega_{S^r})$; on a

$$a_{S^r}(M)=a^{r-1}(S)+a_{D^r}(S)+a^r(M).$$

Dans le cas où M est kählérienne compacte, on définit un espace vectoriel isomorphe à $H^t(M,\Omega^r)/\delta^*H^{t-1}(M,\mathfrak{R}_{S^r})$, d'où une expression de $\dim H^s(M,\Omega_{S^r})$. Résultats analogues à partir du faisceau des germes de r-formes méromorphes fermées multiples de $-S$. Soient S et T deux sous-variétés de dimension $n-1$; S, T et leur intersection $S\cdot T$ sont supposées irréductibles et sans singularité. Considérons le faisceau des germes de r-formes méromorphes multiples de $S+T$; alors,

$$a^r{}_{S+T}(M)=a^{r-1}_{S\cdot T}(S+T)+a^r{}_{S\cdot T+D}(S+T)+a^r(M),$$

où D désigne le diviseur caractéristique de $S+T$.

P. Dolbeault (Paris).

Kodaira, K. On a differential-geometric method in the theory of analytic stacks. Proc. Nat. Acad. Sci. U. S. A. **39**, 1268–1273 (1953).

Soient V une variété kählérienne compacte, de dimension complexe n; F un espace fibré analytique de base V, à fibre vectorielle C de dimension un, de groupe structural le groupe multiplicatif des complexes opérant dans C et $\Omega^p(F)$ le faisceau, sur V, des germes de p-formes holomorphes à coefficients dans F [K. Kodaira, mêmes Proc. **39**, 865–868 (1953); MR **16**, 74]. Si F est défini, sur un recouvrement $\{U_j\}$ de V, par le système de changements de cartes $\{f_{jk}\}$, soit $-F$ l'espace fibré défini par les $\{f_{jk}^{-1}\}$; à partir de l'isomorphisme (1) $H^q(V,\Omega^p(F))\approx H^{p,q}(F)$ (espace vectoriel des formes harmoniques sur V, à coefficients dans F), l'auteur établit l'isomorphisme

$$H^q(V,\Omega^p(F))\approx H^{n-q}(V,\Omega^{n-p}(-F)). \tag{2}$$

On sait [note citée] qu'il existe un système de fonctions réelles positives, de classe C^∞, a_j (définie sur U_j) telles que $a_j/a_k=|f_{jk}|^2$ dans $U_j\cap U_k\neq 0$; alors, si ∂ et $\bar{\partial}$ désignent les parties de type (1, 0) et (0, 1) de l'opérateur de différentiation d, la (1, 1)-forme $\partial\bar{\partial}\log a_j$ définie sur U_j, se prolonge à V et, pour que la classe de cohomologie (au sens du théorème

de de Rham) d'une (1, 1)-forme réelle γ, C^∞, d-fermée soit la classe caractéristique $c(F)$ de F, il faut et il suffit qu'il existe un système de a_j tel que $\gamma=(i/2\pi)\partial\bar\partial \log a_j$. L'auteur associe, à tout point $z \in V$ et à tout tenseur antisymétrique u, de type (p, q), une forme hermitienne $\Theta^{p,q}(\gamma, u; z)$, et, à l'aide d'une méthode de géométrie différentielle due à Bochner [Ann. of Math. (2) 49, 379–390 (1948); 50, 77–93 (1949); MR 9, 618; 10, 571], il montre que, pour toute forme harmonique $\varphi=\{\varphi_j\}$ à coefficients dans F, on a: $\int_V (g/a_j)\Theta^{p,q}(\gamma, \varphi_j; z)\, dV \leqq 0$ pour $q \geqq 1$ (g désigne le déterminant $|g_{\alpha\beta^*}|$ associé à la métrique kählérienne et $dV=(i)^n dz^1 d\bar z^1 \cdots dz^n d\bar z^n$). Il en résulte, compte tenu des isomorphismes (1) et (2), le théorème 1: si $c(F)$ contient une (1, 1)-forme réelle, d-fermée γ telle que $\Theta^{p,q}(\gamma, u; z)$ soit >0 en chaque point $z \in V$, les groupes de cohomologie $H^q(V, \Omega^p(F))$ et $H^{n-q}(V, \Omega^{n-p}(-F))$ sont nuls pour $1 \leqq q \leqq n$. Si $p=n$, la condition $\Theta^{p,q}(\gamma, u; z)>0$ équivaut à: la forme hermitienne dont les coefficients sont ceux de γ est >0, ce que l'on note $\gamma>0$; d'où le théorème 2: si $c(F)$ contient une (1, 1)-forme réelle, d-fermée $\gamma>0$, on a:

$$H^q(V, \Omega^n(F))=0=H^{n-q}(V, \Omega^0(-F))$$

pour $1 \leqq q \leqq n$. Enfin, désignons par espace fibré canonique K, l'espace fibré du type étudié défini par le système des jacobiens des transformations de coordonnées locales de V; on a: $\Omega^0(F) \approx \Omega^n(F-K)$; d'où le théorème 3: si $c(F)-c(K)$ contient une (1, 1)-forme d-fermée réelle $\gamma>0$, alors: $H^q(V, \Omega^0(F))=0$ pour $1 \leqq q \leqq n$. *P. Dolbeault* (Paris).

Akizuki, Yasuo, and Nakano, Shigeo. Note on Kodaira-Spencer's proof of Lefschetz theorems. Proc. Japan Acad. 30, 266–272 (1954).

$\Omega^q(D)$ denotes the faisceau of meromorphic q-forms on an algebraic variety V which are (locally) multiples of the divisor $-D$. Kodaira [see the preceding review] obtained sufficient conditions on D in order that $H^p(V, \Omega^q(D))=0$. This paper obtains similar conditions (not identical with those of Kodaira), showing that $H^p(V, \Omega^q(D))=0$ if $p+q>\dim V$ when there exists a form of type (1-1) dual to D whose coefficients form a positive definite Hermitian matrix. The method depends on a new property of harmonic integrals on a complex line bundle. Several known results in the related field of algebraic geometry are quickly deduced from this result, e.g. Lefschetz's theorem that if S is a generic section of V, then $H^r(V, C) \cong H^r(S, C)$ if $r \leqq \dim V-2$, and the corresponding theorem when $r+1=\dim V$, and also the theorem that a complex line bundle over an algebraic variety is equivalent to a divisor class. *W. V. D. Hodge.*

Differential Geometry

***Julia, Gaston. Cours de géométrie infinitésimale. Quatrième fascicule. Cinématique et géométrie cinématique. Deuxième partie: Etude approfondie du mouvement d'un corps solide.** 2ème éd. Gauthier-Villars, Paris, 1955. 88 pp. 1600 francs.

This second part of Fascicule IV contains Ch. XII on plane cinematics, Ch. XIII on the motion of a solid about a fixed point and Ch. XIV on the general motion of a solid. In plane cinematics distinction is made between properties of the first order (velocities, trajectories) and of the second order (accelerations, curvatures). Stress is laid on the formula of Euler-Savary. Careful attention is paid to the epicycloidal movements with application to the cuspidal epicycloid and the cycloid. As an example of the general motion of a solid we find the motion of the trihedron of Frenet of a space curve. As in previous fascicules [cf. 2ème fasc.; MR 16, 512] there are a number of problems for exercise. *D. J. Struik* (Cambridge, Mass.).

Deaux, R. Sur la spirale conique. Mathesis 63, 328–334 (1954).

Deaux, R. Géodésiques d'un hélicoïde développable ou d'un cône de révolution. Mathesis 63, 363–365 (1954).

Bilinski, Stanko. Einige Eigenschaften sphärischer Evoluten und sphärischer Evolventen. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 9, 109–114 (1954). (Serbo-Croatian summary)

Horninger, H. Über Trochoidenschraublinien und die durch Trochoidenschraubung erzeugbaren Kreisschraubflächen. Monatsh. Math. 58, 193–212 (1954).

Die zur Erzeugung einer ebenen Trochoïde führende Bewegung eines Gelenkparallelograms $FAPB$ wird derart verallgemeinert, dass man A und B auf zwei Schraubenlinien mit gemeinsame Achse f und F auf f führt, wobei die Ebene des Parallelograms zu f normal bleibt: die Bahnkurve von P wird als Trochoidenschraublinie bezeichnet. Sie kann durch zwei verschiedene Zylinderschrotungen gewonnen werden: die Trochoidenschraubungen. Verschiedene Eigenschaften der Kurven werden bewiesen und mittels der genannten Bewegungen auch neue Ergebnisse für gerade Kreisschraubenflächen aufgedeckt, z.B.: die auf einer solchen Fläche befindlichen Schraubenlinien berühren in allen Punkten jedes Kreisschnittes die Erzeugenden einer Schar eines Hyperboloids. *O. Bottema* (Delft).

Biran, Lutfi. Sur le roulement des surfaces reglées. Rev. Fac. Sci. Univ. Istanbul. Ser. A. 19, 61–66 (1954).

Dans ce travail pour étudier le mouvement d'un solide l'auteur fait usage de la notion du vecteur dual de Study et Blaschke. Soient E_1 et E_2 deux solides, chacun animé d'un mouvement hélicoïdal respectivement autour des axes fixes. Si le rapport des rotations et la différence des pas restent constants, les surfaces réglées respectivement base et roulante du mouvement (E_2, E_1) sont des surfaces "à pente duale" constante. *O. Bottema* (Delft).

***Fabricius-Bjerre, Fr. On plane closed curves with two inflectional points.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 42–43 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Let k be a real simple closed plane curve without cusp points and with exactly two inflection points U and V. The latter divide k into two arcs k_1 and k_2. The convex closure of k is bounded by a segment and a convex sub-arc of k_1, say. Then k_1 is a double spiral. If the spiral with the endpoint U [V] meets the straight line UV at p [q] points, k_1 is said to have the characteristic numbers (p, q). In general, k_2 then will be a double spiral with the characteristic numbers $(p-1, q-1)$. The author determines the (real) order and class and the number of double tangents of k. They depend on p, q and the intersections of k with its tangents at U and V. *P. Scherk* (Saskatoon, Sask.).

Demaria, Davide Carlo. I sistemi ∞^6 di curve spaziali godenti della proprietà conforme in prima approssimazione. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 251–262 (1954).

The author studies the family of ∞^6 curves in 3-space which is defined by the system of differential equations $y'''=\varphi$, $z'''=\psi$, where φ and ψ are analytic functions of $x, y, z, y', z', y'', z''$. Through two arbitrary neighboring points P_0: (x_0, y_0, z_0) and P_1: (x_1, y_1, z_1), there pass ∞^2 curves of the family. The condition that the bundles of tangents of these curves at P_0 and P_1 be congruent is expressed by the vanishing of three quantities which, when they are expanded in power series in x_1-x_0, ordinarily begin with terms in $(x_1-x_0)^3$. If the three terms in $(x_1-x_0)^3$ are absent, the author says that the family of ∞^6 curves possesses the conformal property in the first approximation. In this case the variables y'', z'' appear in the functions φ, ψ in a certain specific way, which is exhibited. A few other properties of families of curves possessing the above property are considered. In particular, the author discusses the geometrical significances of certain natural restrictions on the way in which y' and z' occur in φ and ψ.

L. A. MacColl (New York, N. Y.).

Ascoli, Guido. Sull'equazione integrale da cui dipende la ricerca di una curva gobba di cui siano note le curvature in funzione dell'arco. Matematiche, Catania **8**, no. 2, 11–18 (1953).

The conditions are here determined under which the Volterra equation given by Nalli [Ann. Mat. Pura Appl. (4) **17**, 193–202 (1938)] for the study of a twisted space curve yields the case of the "closed cycle" (ciclo chiuso), finding again results obtained by Sorace [Mathematiche, Catania **2**, 65–79 (1947); Rend. Accad. Sci. Fis. Mat. Napoli (4) **14**, 155–166 (1948); Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 1, 55–73 (1948); MR **10**, 189; **14**, 404; **11**, 616]. The results are established by using the method of the Laplace transform. It is found that the only curves satisfying the required conditions are the circle, the circular helix, and the evolutes of the circular helix. *S. B. Jackson.*

Sternberg, Shlomo. Legendre transformations of curves. Proc. Amer. Math. Soc. **5**, 942–945 (1954).

Let $y(x)$ be a convex bounded function on the interval (a, b). Let j be the set of points x such that $y_-'(x)\neq y_+'(x)$, and let k be the set of the closures of the x intervals where $y_+'(x)$ (or $y_-'(x)$) is constant. Let K be the set of closures of the intervals of y'-values satisfying $y_-'(x)<y'<y_+'(x)$ for some x on j, and J the set of y'-values for which $y'=y'(x)$ for some x on k. Let $A=y_+'(a)$ and $B=y_-'(b)$.

Consider the correspondance $X=y'(x)$, which maps $(a, b)-(j+k)$ onto $(A, B)-(J+K)$. Let $T(X)$ be the set of those x-values at which $y_-'(x)\leqq X$, and let

$$\bar{x}=\bar{x}(X)=\sup T(X).$$

Define $Y=Y(X)$ as $Y=\bar{x}(X)\cdot X-y(\bar{x}(X))$. Letting $Y'(X)$ be the derivative of Y with respect to X, the author proves that: Y_+' and Y_-' exist everywhere and are nondecreasing with $Y_-'\leqq Y_+'$ and $Y_+'(X)=\bar{x}(X)$. Furthermore, $Y_-'\neq Y_+'$ only on J and Y_+' is constant only on intervals of K.

The result is applied to show that the (outside) parallel curves of a convex curve are of class C^1 (at least). It also shows the existence of a non-linear solution of $y=y'x-f(y')$ when f is strictly convex. *C. B. Allendoerfer.*

Muracchini, Luigi. Sulle trasformazioni puntuali fra piani proiettivi sovrapposti. Boll. Un. Mat. Ital. (3) **9**, 360–366 (1954).

The author initiates a study of analytic point transformations in a single plane, from the point of view of projective differential geometry, and obtains local and global differential invariants of such a transformation. The global invariants are obtained by the use of E. Cartan's method of a moving frame of reference. *J. A. Todd.*

Nicolescu, Alexandru. Un invariant projectif. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 225–234 (1954). (Romanian. Russian and French summaries)

When T_1 and T_2 are arbitrary points on the plane curves C_1 and C_2, γ_1 and γ_2 the radii of curvature to C_1 and C_2 at T_1 and T_2, and M the intersection of the tangents at T_1 and T_2, then

$$I_a=\frac{MT_1^3}{\gamma_1}:\frac{MT_2^3}{\gamma_2}$$

is a projective invariant, which for C_1 and C_2 tangent at $T=T_1=T_2$ and M on the common tangent passes into the invariant of Mehmke and H. J. S. Smith. $C_1=C_2$, $I_a=1$ gives the conic sections (Demoulin). It is shown that every symmetric function of I_a and $1/I_a$ is an absolute invariant of the general projective group in the plane. A method is developed to construct differential invariants of this transformation. It is also shown that the group G of one-to-one, analytic transformations which leave a differential equation of the second order invariant is of the form $G=TPT^{-1}$, where T is a contact transformation and P the group of point transformations which are one-to-one and bicontinuous.

D. J. Struik (Cambridge, Mass.).

Myller, A. Courbes et surfaces parallèles au sens large. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **5**, 1–15 (1954). (Romanian. Russian and French summaries)

The equations of a plane curve $x=f(p)$, $y=g(p)$ are said to be in clinoidal form if $p=dy/dx$. In this case a function $F(p)$ exists such that $f(p)=F'(p)$, $g(p)=pF'(p)-F(p)$. From two curves with equations in clinoidal form $x=f_1(p)$, $y=g_1(p)$; $x=f_2(p)$, $y=g_2(p)$ the family of curves

$$x=\frac{f_1(p)+tf_2(p)}{1+t},\quad y=\frac{g_1(p)+tg_2(p)}{1+t}$$

is constructed, where t is a parameter. For all these curves $dy/dx=p$. This is a family of curves with the property that the tangents at the points of intersection with each of the lines of a given family of straight lines are parallel. Such a family is a generalization of the families of parallel curves in the sense of Leibniz. Some properties of such families are derived. Generalization to space gives families of surfaces of which the tangent planes at the points of intersection with each of the lines of a given congruence of straight lines are parallel. A previous article on correspondence by parallel tangents, written by the author, can be found in the Ann. Sci. Univ. Jassy **18**, 1–6 (1931). *D. J. Struik.*

Myller, A. L'équation arco-radiale de Sylvester. Acad. Repub. Pop. Române. Fil. Iaşi. Stud. Cerc. Şti. **4**, 19–28 (1953). (Romanian. Russian and French summaries)

Sylvester used arco-radial equations, $s=f(r)$, to study plane curves. [Rep. 38th Meeting British Assoc. Adv. Sci.,

Norwich, 1868, Notices and Abstracts, pp. 10–11 = Collected Math. Papers, vol. 2, Cambridge, 1908, p. 629]. If one studies curves on a surface S, then the equation $s=f(r)$ determines a network of curves on S. Call the plane curve determined by $s=f(r)$ the profile curve. If a cone is formed with vertex at the origin and a curve C of the network on S as directrix, and if this cone is applied to a plane, then the curve C becomes precisely the profile curve. Also, an arco-radial network is a network without detours. Several special results are obtained. For instance, let spherical level lines on S be the curves of intersection of S with spheres with center at 0. All curves of an arco-radial network, when meeting a spherical level line, form with the radius vector at the point of intersection the same angle $V=\arccos 1/f'(r)$. Again, curves on a surface for which the distance from a tangent line to the origin is a given function of the radius vector of the point of contact can be studied as an arco-radial network. Finally, the differential equation of an arco-radial network is integrated in the case of a surface of revolution and in one other case; in the former case the author finds surfaces for which the arco-radial curves will be loxodromes. *A. Schwartz* (New York, N. Y.).

Myller, A. Chemins de profil longitudinal donné. Acad. Repub. Pop. Române. Fil. Iaşi. Stud. Cerc. Şti. **4**, 1–17 (1953). (Romanian. Russian and French summaries)

Given a surface S and a plane curve C called a profile indicatrix curve. The problem is to find the paths on S whose longitudinal profiles are precisely C; i.e., if the cylinder which is the vertical projection of such a path is applied to a plane, the path on S becomes precisely the plane curve C. The author finds the differential equation for these paths, and points out that their projections in the xy-plane form a network of paths without detour. [Cf. G. Scheffers, Jber. Deutsch. Math. Verein. **16**, 421–422 (1907).] A section is devoted to networks of paths of given profile such that pairs of paths which intersect on the same level line have xy-plane projections intersecting at the same angle. Another section is devoted to the special cases where the surface S has level curves which i) are curves of translation; ii) are conoidal, i.e., one can find a point P on the map of level curves such that two level curves cut segments of the same length from all radius vectors drawn through P; iii) are parallel. A last section is devoted to the inverse problem: Given a plane indicatrix curve C and a family F of curves in the xy-plane, find a surface S for which the family F will be the xy-plane projection of a family of curves on S of the given profile, and find the second family of curves of the given profile. *A. Schwartz* (New York, N. Y.).

Hammer, Preston C. Areas swept out by tangent line segments. Math. Mag. **28**, 65–70 (1954).

Let $\mathbf{x}=\mathbf{x}(s)$ be the vector equation of a twice continuously differentiable skew curve C; $|\mathbf{x}'(s)|=1$; $|\mathbf{x}''(s)|>0$. Thus $\alpha=\int_0^s|\mathbf{x}''(s)|\,ds$ can be used as a parameter on C. Let D: $\mathbf{z}=\mathbf{z}(s,\rho)=\mathbf{x}(s)+\rho\mathbf{x}'(s)$ be the tangential developable of C. The transformation mapping the point $\mathbf{z}$ of D onto the point in plane with the polar coordinates ρ, α is area-preserving. This is a corollary of the following remark: If $\rho(s)\geqq 0$ is a continuous function, the area on D between C and the curve $\mathbf{z}(s,\rho(s))$ is equal to $\frac{1}{2}\int_0^\alpha \rho^2 d\alpha$. This integral is independent of C if ρ is given as a function of α. Various other applications. *P. Scherk* (Saskatoon, Sask.).

Wunderlich, Walter. Beitrag zur Kenntnis der Minimalspiralflächen. Rend. Mat. e Appl. (5) **14**, 1–15 (1954).

The spiral surfaces [$=S$] and the minimal S's are briefly discussed by Darboux [Leçons sur la théorie générale des surfaces, vol. I, Gauthier-Villars, Paris, 1887, pp. 107 ff., 307 ff.]. In the present paper (i) a class of translation S's is constructed which contains all the minimal S's. (ii) The classical theory of minimal surfaces is reviewed from a more geometrical point of view. (iii) This theory is applied to the minimal S's and to the families of associate minimal S's. The results are somewhat similar to those on minimal helicoids. The paper contains excellent drawings [It would be of interest to know whether the class of S's mentioned under (i) actually consists of all the translation S's]. *P. Scherk* (Saskatoon, Sask.).

Grigor'ev, I. N. An asymptotic transformation of p-orthogonal-conjugate systems in n-dimensional space. Dokl. Akad. Nauk SSSR (N.S.) **97**, 765–767 (1954). (Russian)

This investigation generalizes Bianchi's study of triply orthogonal systems with one family of surfaces of constant negative curvature and which admit a two-parametric family of transformations. In euclidean n-space E_n a p-orthogonal system ($p\leqq n$) is defined as a p-dimensional surface consisting of p one-parametric families of $(p-1)$-dimensional surfaces intersecting in mutually orthogonal lines. For $p<n$ there are such systems for which orthogonality of intersection does not mean that the lines are also conjugate, as the theorem of Dupin demands for $p=n$. The properties of being orthogonal and conjugate have to be both postulated. Certain transformations of p-orthogonal-conjugate systems of E_n into others of the same kind are now defined as asymptotic transformations. It is shown that the class of p-orthogonal-conjugate systems which allow such transformations consists of $(p-3)$-parametric systems of congruent pseudospherical triply orthogonal systems each lying in an E_3. Every such a triply orthogonal element is transformed by means of pseudospherical congruences into an element of the same kind in the same E_3. *D. J. Struik* (Cambridge, Mass.).

Haimovici, Adolf. Sur la géométrie d'une équation de Monge de type particulier. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **5**, 17–27 (1954). (Romanian. Russian and French summaries)

The general theory of Monge equations

$$F(x, y, z, dx, dy, dz)=0$$

has been outlined by D. Sintzow [Trudy Sem. Vektor. Tenzor. Anal. **4**, 182–183 (1937)]. Here the case is discussed of an equation $a_{ijk}dx^i dx^j dx^k=0$ in a Riemannian V_3, with the condition that the cone defined by the equation degenerates into three planes, which is the case of the nets of curves. In every plane tangent to two curves of the net there are two asymptotic directions, and two directions tangent to the lines of curvature of the first kind as well as two directions tangent to the lines of curvature of the second kind. There is also a difference between geodesics of the first and second kind, the first defined by means of the osculating plane, the second by means of the extremal property. Both types coincide when the nets are surface building, when also the two types of curvature lines coincide. *D. J. Struik* (Cambridge, Mass.).

Marcus, F. Sur quelques propriétés de congruences de droites et sur leurs représentations sphériques. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **5**, 29–38 (1954). (Romanian. Russian and French summaries)

The focal points F_i, $i=1, 2$, of a line congruence describe curves on the developables of this congruence; the traces of these developables on the spherical image of the congruence are taken as the parametric curves (u, v) on the sphere, whose element becomes $ds'^2=E\,du^2+2F\,dudv+G\,dv^2$. This is the setting for the work on line congruences by Bianchi [Lezioni di geometria differenziale, v. I, 3d ed., Spoerri, Pisa, 1922, p. 483]. From the Bianchi formulas the author derives some new results. Among them are the theorems: 1) The geodesic curvatures of the curves $u=$const., $v=$const. are equal and of opposite sign to the ratios of torsion and curvature of the curves described by the points F_i. 2) The ratio of these geodesic curvatures is equal to the ratio of the area elements of the spherical images referred to the focal surfaces of the congruence. 3) The geodesic curvatures of the spherical images of the asymptotic lines on a surface are bending invariants, and so is the product of the geodesic curvature of the spherical images of the lines of curvature. Reference is made to Gh. Ţiţeica [Acad. Roum. Bull. Sect. Sci. **18**, 109–112 (1937)]. *D. J. Struik.*

Vincensini, Paul. Sur une traduction métrique de l'existence des quadriques de Lie d'une surface. C. R. Acad. Sci. Paris **240**, 481–483 (1955).

If we take the two focal surfaces of a normal line congruence (C), then the planes of osculation at two homologous points on these surfaces to the curves of two focal nets (which are not edges of regression of the developables of (C)), are orthogonal. One of the ways of proving this theorem is by means of the recently published representation of lines in euclidean E_3 by points in euclidean E_4 [see P. Vincensini, C. R. Acad. Sci. Paris **239**, 1113– 1114 (1954); MR **16**, 400]. Then it is also seen that in these osculation planes there exist two mutually focal conics with a contact of the second order with these curves of the focal nets. Both properties (orthogonality of planes and existence of focal conics) characterize normal congruences. *D. J. Struik.*

Backes, F. Sur la courbure géodésique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 1080–1089 (1954).

La formule classique de Bonnet donnant la courbure géodésique d'un courbe tracée sur une surface et définie par $\varphi(u, v)=0$ et celle de Gauss-Bonnet sont démontrées en suivant une méthode qui diffère peu de la voie classique. L'exposé se termine par une application aux membranes liquides en équilibre, affectant la forme d'une surface minima. *F. Şemin* (Istanbul).

Santaló, L. A. Some generalizations of the four-vertex theorem. Math. Notae **12–13**, 69–78 (1954). (Spanish)

Using the method of Herglotz, proof of the four-vertex theorem, generalizations of this theorem and analogues of them are obtained in euclidean, affine, or projective planes or 3-spaces. It may be sufficient to review the case of the euclidean plane. Let C: $\xi=\xi(s)$ be a closed convex curve of length L; $\xi'^2(s)=1$; $\kappa(s)=\sqrt{\xi''^2(s)}>0$. Let $\beta(s)$ be a twice differentiable function of period L. Then the function $(\kappa^{-1}\beta')'+\beta\kappa$ has not less than four extremums on C. A special case of this result yields a generalization of the four-vertex theorem due to Blaschke: Suppose a second closed convex curve is mapped on C such that corresponding points have parallel oriented tangents. Then the quotient of their curvatures has at least four extremums. [References: W. Blaschke, Vorlesungen über Differentialgeometrie, vol. I, 3rd. ed., p. 31, vol. II, 1st and 2nd ed., p. 65, Springer, Berlin, 1930, 1923.] *P. Scherk* (Saskatoon, Sask.).

Chern, Shiing-shen, Hartman, Philip, and Wintner, Aurel. On isothermic coordinates. Comment. Math. Helv. **28**, 301–309 (1954).

Hartman et Wintner [Amer. J. Math. **75**, 260–276 (1953); MR **14**, 1015] ont établi le résultat suivant: soit $S\,[X=X(u, v)]$ un morceau de surface de classe C^n $(n\geqq 3)$; il existe une paramétrisation $X=X(U, V)$ de S pour laquelle $X(U, V)$ est de classe C^n et qui donne au ds^2 de S la forme normale conforme $ds^2=\gamma(dU^2+dV^2)$. Le but de ce papier est d'étudier le résultat analogue pour $n=2$. Le théorème principal est le suivant: étant donnée, sur le domaine $u^2+v^2<r^2$, une métrique définie positive ds^2 où les $g_{ik}(u, v)$ sont de classe C^1 et qui admet une courbure continue $K=K(u, v)$, il existe une application de classe C^1, à jacobien non nul, qui donne au ds^2 la forme normale conforme. Toute application admettant ces propriétés est de classe C^2, si bien que $\gamma(U, V)$ est de classe C^1.

La courbure K est entendue ici au sens de Weyl [Vierteljschr. Naturf. Ges. Zürich **61**, 40–72 (1916)]. Si le ds^2 est rapporté à des corepères orthonormés $(ds^2=\omega_1^2+\omega_2^2)$, soit ω_{12} la forme de connexion riemannienne (de classe C^1). La métrique admet une courbure continu si la forme ω_{12} est régulière, c'est-à-dire, satisfait à une formule de Stokes; si $d\omega_{12}$ est l'intégrand pour cette formule, $d\omega_{12}=-k\omega_1\wedge\omega_2$ et k est la courbure. On sait qu'on peut trouver des métriques à coefficients continus pour lesquelles il n'existe aucune transformation de classe C^1, à jacobien non nul, les ramenant à la forme normale conforme. Dans un appendice les A. montrent qu'il existe de telles métriques qui sont celles d'une surface de classe C^1. *A. Lichnerowicz* (Paris).

Bakel'man, I. Ya. Determination of a smooth surface by first and generalized second quadratic forms. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 155–162 (1954). (Russian)

Let D be a domain in a (u, v)-plane bounded by a simple closed polygon. Consider the class W of all surfaces in E^3 given by position vectors $\mathfrak{r}(u, v)$ defined in D, where $\mathfrak{r}(u, v)$ is of class C^1, $|\mathfrak{r}_u\times\mathfrak{r}_v|>0$, and all generalised second partial derivatives of $\mathfrak{r}(u, v)$ in the sense of Sobolev exist. Let E, F, G, L, M, N be defined in the usual way, where, for instance, $\mathfrak{r}_{uu}$ in $L=|\mathfrak{r}_{uu}, \mathfrak{r}_u, \mathfrak{r}_v|$ stands for a generalized derivative. If for two surfaces in W a one-to-one correspondence between their points exists such that the values of E, F, G, L, M, N at corresponding points are equal almost everywhere in D, then the two surfaces are congruent as sets in E^3.

H. Busemann (Paris).

Manara, Carlo Felice. Caratterizzazione integrale di certe superfici immerse in varietà riemanniane tridimensionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 15–21 (1954).

Consider a 3-dimensional Riemannian variety, V_3. For a curve in V_3 there can be defined a quantity T which may be called the torsion since it reduces to the ordinary torsion when V_3 is Euclidean 3-space. If Σ is a surface in V_3, it is shown that a necessary and sufficient condition that $\int_\Sigma T\,ds=0$ for any closed curve on Σ is that Σ have indeterminate lines of curvature. *S. B. Jackson.*

Schwartz, Marie-Hélène. Formules apparentées à la formule de Gauss-Bonnet pour certaines applications d'une variété à n dimensions dans une autre. Acta Math. **91**, 189–244 (1954).

Let V be a manifold of n dimensions ($n>1$) of class C^2 and orientable and compact. Let W be a manifold of the same type, but not necessarily compact. A mapping $f: V\to W$ is called regular if (1) f is of class C^2, (2) there exists a regular triangulation, D, of V such that the restriction of f to each open simplex of D is a bidifferentiable homeomorphism of class C^2 onto a differentiable subspace of W. The paper discusses a generalization of the Gauss-Bonnet formula which applies to such mappings, and is closely related to the work of Ahlfors [Acta Soc. Sci. Fenn. Nova Ser. A. **2**, no. 6 (1937)].

Given the triangulation D of V, a special vector for D is a non-null vector defined at $x\in V$ and tangent to one of the simplices of D which contains x. Let $\mathfrak{V}$ be the fibre space of non-null tangent vectors on V, $\mathfrak{V}_s$ the fibre space of special vectors on V, V^+ the fibre space of all tangent vectors on V, and $\mathfrak{V}_s^+$ the fibre space of all special or null vectors on V. Similar notations apply to W. The mapping f then induces a mapping $f: \mathfrak{V}_s^+\to\mathfrak{W}^+$.

Suppose that W has a Riemannian metric and let forms Ω_0 be defined on W, and Ω and Π on $\mathfrak{W}$ such that $\int\Omega_0=\chi(W)$, $\Omega=\pi^*\Omega_0$, where π maps each vector into its origin, and $\Omega=-d\Pi$. These forms are known to exist from Chern's proof of the Gauss-Bonnet formula. Define Ω_0^* on V as $f^*(\Omega_0)$, Ω^* and Π^* on $\mathfrak{V}_s$ as $\bar{f}^*(\Omega)$ and $\bar{f}^*(\Pi)$ respectively. Then $\Omega^*=-d\Pi^*$.

The main results are the following. (1) If C is an n-dimensional transversal subvariety of V

$$\int_C\Omega_0^*=\int_{\mathfrak{R}^+(\dot{C})}-\Pi^*+\chi_f(C),$$

where $\mathfrak{R}^+(\dot{C})$ is a cycle in $\mathfrak{V}_s$ induced by $\dot{C}$, the boundary of C, and $\chi_f(C)=\sum_{i,k}(-1)^k m(D_i'^k)$ for $D_i'^k\subset C$, where $D_i'^k$ are simplices of a subdivision of D and $m(D_i'^k)$ is the topological degree of f on $D_i'^k$. (2) If U is a transversal subvariety of dimension $\leqq n-1$,

$$\int_{\mathfrak{R}(U)}\Pi=\chi_f(U).$$

(3) If U is a transversal subvariety of uneven dimension $\leqq n$,

$$\chi_f(\dot{U})=2\chi_f(U).$$

C. B. Allendoerfer (Seattle, Wash.).

Hlavatý, V. Embedding theory of a W_m in a W_n. II. Rend. Circ. Mat. Palermo (2) **3**, 149–192 (1954).

This paper is a continuation of a previous paper by the author [same Rend. (2) **1**, 403–438 (1953); MR **15**, 349]. The integrability conditions of the Frenet equations of a W_m in a W_n (Weyl space) are discussed. They give some information about the possibility of embedding a W_m in W_n. The second part of the paper deals with the geometry of a W_p in a W_m in a W_n. Relations are given between the F-tensors of curvature of W_p in W_m, W_m in W_n and W_p in W_n. It is shown that in the case $p=n-2$, $m=n-1$ many classical results for $n=3$ in metric geometry may be generalized. In the last section the relations to other papers on embedding theory are discussed. *J. Haantjes* (Leiden).

Teleman, C. Les groupes transitifs de mouvement des espaces de Riemann V_5. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 503–526 (1953). (Romanian. Russian and French summaries)

Where G. Vrănceanu [same Stud. Cerc. Mat. **4**, 121–153 (1953); MR **16**, 72] has studied the motion groups in a Riemannian V_4, the present author continues this work for V_5 with transitive groups of motion. The results of S. Medici are verified and it is shown that there exists one non-separable group in the Euclidean E_5, which is the G_2 given by

$$2X_{12}+X_{34},\quad X_{35}+\frac{1}{\sqrt{3}}(X_{13}+X_{24}),\quad X_{15}+\frac{1}{\sqrt{3}}(X_{23}-X_{14}),$$

where $X_{ij}=\partial_j f_i-\partial_i f_j$. The other groups in E_5 are separable, and can be given as follows:

G_1: $X_{12}+\lambda X_{34}$; G_2: X_{12}, X_{34}; G_3: $X_{12}-X_{34}$, $X_{12}+X_{34}$, $X_{14}-X_{23}$; G_3': X_{12}, X_{35}, X_{15}; G_4: X_{12}, X_{34}, $X_{12}+X_{34}$, $X_{14}-X_{23}$; G_4': X_{12}, X_{15}, X_{25}, X_{34}; G_6: X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}.

Among the other results we find the ds^2 for the V_5 with a transitive motion group G_9 with the stability group G_4 mentioned above, and the V_5 with transitive group G_7 and the stability group G_3 mentioned above. *D. J. Struik.*

Vrănceanu, G. Groupes de mouvement des espaces à connexion. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 387–444 (1951). (Romanian. Russian and French summaries)

An affine space A_n admits a group of transformations into itself (automorphisms) of at most n^2+n parameters, if curvature and torsion are zero. If curvature or torsion are not zero, then Egorov's theorem states that the number of parameters is at most n^2. The maximum is reached only for spaces with an invariant Pfaffian form; if A_n is without torsion it then is projectively euclidean, so that the coefficients of connection can be written in the form $\Gamma^i_{jk}=-\delta_j{}^i\varphi_k-\delta_k{}^i\varphi_j$, where ϕ_i is a covariant vector. In the case of n^2 parameters the group is either of the form $x'^1=x^1+a^1$, $x'^i=a_j{}^i x^j+a^i$, $i=2, 3, \cdots, n$, or of the form $x'^1=ax^1$, $x'^i=a_j{}^i x^j+a^i$. In the first case $\Gamma^1_{11}=-2$, $\Gamma^k_{k1}=-1$, $\Gamma^k_{1k1}=1$ (no summation), in the second case $\Gamma^1_{11}=-2k/x^1$, $\Gamma^k_{k1}=-k/x^1$, $\Gamma^k_{1k1}=k^2/(x^1)^2$; Γ^r_{jkl} is the curvature tensor. Since all components of the derivative of this tensor are zero except the $\Gamma^k_{1k1,1}$, the spaces are not symmetrical. In the case when the A_n is not projectively euclidean the number of parameters of the group of transformations into itself is at most n^2-2n+5, a theorem which can be proved by selecting a coordinate system in which $\gamma^1_{223}\neq 0$, γ^α_{223}, $\alpha\neq 1$, the γ^a_{bcd} represents the curvature tensor with respect to a suitable system of congruences. The maximum n^2-2n+5 is reached in the case of an A_n in which it is possible to reduce all Γ^i_{jk} to zero except $\Gamma^1_{22}=x^2$, which admits the group

$$x'^1=b^2cx^1+\frac{b^2a}{6}(x^2)^3+\frac{b^2\gamma}{2}(x^2)^2+a_j{}^i x_i+a,$$

$$x'^2=bx^2+\beta,\quad x'^3=cx^3+ax^2+\gamma,$$

$$x'^i=a_j{}^i x^j+a^i,\quad i=4, \cdots, n,\ j=2, \cdots, n.$$

If the A_n has a component $\gamma^i_{234}\neq 0$ ($n\geqq 4$) and if the γ^k_{hkl}, $l, h, k=1, 2, 3, 4$, are zero, then the group has at most n^2-3n+8 parameters, and this maximum is reached.

As to the projective P_n, they have the same maximum numbers as the A_n which are not projectively euclidean; hence n^2-2n+5 and n^2-3n+8. When an A_n has nonzero torsion and one of the anholonomic torsion coefficients $t^a{}_{bc} \neq 0$ for $a \neq b, c$, then the group has at most n^2-2n+6 parameters.

The author [Leçons de géométrie différentielle, v. I, Bucarest, 1945; MR 9, 532] has shown that a reducible V_n, that is, a V_n to which an invariant form of rank m, $p=(ds^1)^2+\cdots+(ds^m)^2$ can be associated ($1 \leqq m < n$), admits a group of automorphisms with at most $n+\frac{1}{2}(m-1)+\frac{1}{2}(n-m)(n-m-1)$ parameters. The maximum is only reached in the case that p and its complementary form q ($ds^2=p+q$) are of constant curvature. Egorov's theorem is a special case for $m=1$. For $m=2$ the maximum is reached for a reducible V_n which is not conformally euclidean. In the Einstein case the maximum $\frac{1}{2}(n-1)(n-2)+2$ is reached for $n=4$ in the case of the symmetric space for which

$$\gamma^1{}_{313}=\gamma^1{}_{414}=\lambda,\ \gamma^1{}_{212}=4\lambda,\ \gamma^h{}_{kkh}=\gamma^\alpha{}_{\beta\alpha\beta} \quad (h\neq k\neq\alpha\neq\beta),$$
$$\gamma^1{}_{224}=2\lambda,\ \gamma^1{}_{342}=\gamma^1{}_{423}=-\lambda,\ \lambda=\text{const.}$$

When $n>4$ an Einstein V_n admits at most a group of $\frac{1}{2}(n-1)(n-2)+5=M$ parameters, and the only possibilities are M, $M-1$, $M-2$; M can only be reached for $n<7$. Expressions for the $\gamma^a{}_{bcd}$ in the case of maximum M, $n=4$ ($M=8$) and maximum M, $n=6$ are given; in both cases the spaces are symmetrical. When a V_n ($n \geqq 7$) is irreducible, then its group has at most $M-1$ parameters; if n is even this maximum is at most $n+(\frac{1}{2}n)^2$. [For the literature see G. Vrănceanu, C. R. Acad. Sci. Paris **229**, 336–338, 543–545 (1949); Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 121–153 (1953); MR **11**, 134; **16**, 72; I. P. Egorov, Dokl. Akad. Nauk SSSR (N.S.) **73**, 265–267 (1950); MR **12**, 636.] *D. J. Struik.*

Teleman, C. Sur les groupes maximums de mouvement des espaces de Riemann V_n. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 143–171 (1954). (Romanian. Russian and French summaries)

This study is closely related to the investigations of G. Vrănceanu [see, e.g., same Stud. Cerc. Mat. **4**, 121–153 (1953); MR **16**, 72; and the preceding review; see also the paper reviewed below]. It is shown that every subgroup G_r of the complete group of rotations in n variables is a motion group of a space V_{n-1} of constant positive curvature which keeps two or more complementary systems of Pfaff invariant. Then the theorem is found that a group G_r of rotations in n variables, real and irreducible, with $r<n(n-1)/2$ parameters, has at most p^2 parameters if $n=2p$ or $n=2p+1$. Hence the spaces $V_{2p}(\lambda)$ of Vrănceanu are those irreducible Riemannian V_{2p} of variable curvature which have a motion group of the maximum number of parameters, p^2+2p. A $V_{2p}(\lambda)$ is defined as a Riemannian V_{2p} with a transitive group of motions and the group

$$X_{2r-1,2r},\quad X_{2s-1,2r-1}+X_{2s,2r},\quad X_{2s-1,2r}-X_{2s,2r-1},$$
$$X_{ij}=x_i\partial_j f-x_j\partial_i f$$

as group of stability. At the conclusion of the paper we find an expression for the metric of a V_{2p+1}, irreducible and of variable curvature, which has a transitive motion group with the maximum of parameters, $(p+1)^2$. It is shown how such $V_{2p}(\lambda)$ can be realized on a hypersphere in euclidean $(2p+2)$-space [see the paper reviewed below].

D. J. Struik (Cambridge, Mass.).

Vrănceanu, G. Sur une classe d'espaces riemanniens homogènes. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 173–223 (1954). (Romanian. Russian and French summaries)

The newer study of Riemannian V_n with transformations in themselves starts with E. Cartan [e.g., Bull. Soc. Math. France **54**, 214–265 (1926); **55**, 114–134 (1927); J. Math. Pures Appl. (9) **8**, 1–33 (1929); Ann. Soc. Polon. Math. **8**, 181–225 (1930); for further references see G. Vrănceanu, Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 121–153 (1953); MR **16**, 72]. In the present paper we deal with certain V_{2p} which admit a transitive group of motions of $2p+p^2$ parameters. The stability group $G_p{}^2$ of these V_{2p} admits a particular transformation $X_{12}+X_{34}+\cdots+X_{2p-1,2p}$, where $X_{pq}=\partial_p f_q-\partial_q f_p$, and is the largest orthogonal group of this property. The V_{2p} has thus the maximum group of motions among the spaces of which the group of stability has this particular transformation. The group $G_p{}^2$ is closed and is composed of this particular transformation and a simple $G_p{}^2{}_{-1}$. Hence the group G_{2p+p^2} is simple, and is closed. Then follow investigations into the form of the metric of these spaces and the equations of their geodesics. Teleman, in the paper reviewed above, has shown that a $V_{2p}(\lambda)$ can be considered as a non-holonomous manifold on the spheres S_{2p+1} in a euclidean E_{2p+2}. The author shows how this immersion can be explicitly realized. The $V_{2p}(\lambda)$ thus obtain a metric

$$ds^2=\frac{(dx^1)^2+\cdots+(dx^{2p})^2}{u}-k\frac{(x^\lambda dx^\lambda)^2+w^2}{u^2},$$
$$u=1+k[(x^1)^2+\cdots+(x^{2p})^2],$$
$$w=x^1dx^2-x^2dx^1+\cdots+x^{2p-1}dx^{2p}-x^{2p}dx^{2p-1}$$
$$k=4\lambda=\text{const},\quad h=1,\cdots,p.$$

D. J. Struik (Cambridge, Mass.).

Petrescu, Şt. La classification des espaces à connexion affine A_2. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 322–363 (1951). (Romanian. Russian and French summaries)

A number of theorems are derived concerning A_2 (with and without torsion) which have a group of transformations of congruences which admits a form of Pfaff which is not exact. The group of transformations is taken in the form $d\bar{s}^1=ds$, $d\bar{s}^2=\alpha ds^1+ds^2$, $\alpha=\alpha(x^1,x^2)$. It is shown that it is possible to cast the connection of such an A_2 with torsion into the form

$$\Gamma^1{}_{11}=(\mu+c)e^{s^2},\quad \Gamma^1{}_{12}=\Gamma^1{}_{21}=-\tfrac{1}{2},\quad \Gamma^1{}_{22}=0$$
$$\Gamma^2{}_{11}=\tfrac{1}{2}(c^2+\rho)e^{2s^2},\quad \Gamma^2{}_{22}=-\tfrac{1}{2},\quad \Gamma^2{}_{12}=ce^{s^2},\quad \Gamma^2{}_{21}=\mu e^{s^2},$$

where μ and ρ are invariants and c is constant. Several special cases are discussed. Reference is made to G. Vrănceanu [Leçons de géométrie différentielle, v. I, Bucarest, 1947; Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 813–821 (1949); MR **9**, 532; **14**, 1123] as well as to A. E. Liber [Mat. Sb. N.S. **27(69)**, 249–266 (1950); MR **13**, 73].

D. J. Struik (Cambridge, Mass.).

Petrescu, Şt. Considérations concernant les espaces à connexion projective P_2. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **3**, 529–558 (1952). (Romanian. Russian and French summaries)

The methods developed by the author in the paper reviewed above are applied to projective P_2 with torsion but without curvature. A classification of such P_2 is given with

respect to the transformations of congruences which they admit. Special attention is paid to the case where also the torsion is zero. *D. J. Struik* (Cambridge, Mass.).

Petrescu, Ş̧t. Classification des espaces P_3. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 453–502 (1953). (Romanian. Russian and French summaries)

This paper is a continuation of the one reviewed above, and deals with the case where torsion and curvature are not zero. The classification is according to the groups of projective transformations of congruences $ds^a=c_b{}^a(x^1, x^2)ds^b$, $d\delta^0=ds^0+c_b{}^0(x^1, x^2)ds^b$ which these spaces admit. Canonical forms are developed. Special cases are those of zero affine torsion but projective torsion and curvature $\neq 0$, and those of torsion zero. *D. J. Struik* (Cambridge, Mass.).

Couty, Raymond. Sur les espaces symétriques harmoniques de Walker. C. R. Acad. Sci. Paris **239**, 1576–1577 (1954).

This paper gives 2-forms with zero covariant derivatives for each of the symmetric harmonic Riemannian 4-spaces constructed by the reviewer [J. London Math. Soc. **24**, 21–28 (1949); MR **10**, 739]. One of these spaces is identified with the homogeneous space $SU(3)/U(2)$.

A. G. Walker (Liverpool).

Lelong-Ferrand, Jacqueline. Formes différentielles définies sur une variété admettant un groupe continu d'isométries. C. R. Acad. Sci. Paris **240**, 268–269 (1955).

It is known that if a Riemannian variety V_n admits a group of isometries G, then the harmonic forms on V_n are invariant for the infinitesimal group g of G when V_n is compact. In this note are some generalizations of this result to the case in which V_n is not compact, and ∂V_n is invariant. (1) If ϕ is harmonic and $\phi=0$ on δV_n, ϕ is invariant; (2) if g is semi-simple, or if ϕ is a form with bounded norm, then if all the kth Lie derivatives of ϕ (for any fixed k) vanish, then ϕ is invariant; (3) every limit problem associated with the equation $\Delta\phi=T$, in which the data are invariant, has all its bounded solutions invariant, provided that the associated homogeneous equation has only harmonic solutions.

W. V. D. Hodge (Cambridge, England).

Tomonaga, Yasuro. Note on Betti numbers of Riemannian manifolds. III. J. Math. Soc. Japan **6**, 37–39 (1954).

L'A. complète certains de ses résultats [même J. **5**, 59–64, 65–69 (1953); MR **15**, 828] concernant les vecteurs de Killing et les formes harmoniques des variétés riemanniennes compactes orientables, dans la voie des travaux de Bochner et du rapporteur. En ce qui concerne les vecteurs de Killing, il introduit une forme quadratique définie positive de coefficients A_{ij} et suppose que la forme quadratique de coefficients

$$\tfrac{1}{2}\Delta A_{ij}+A_{ia}R_j{}^a$$

est soit définie, soit semi-définie négative. Dans le premier cas, la variété n'admet pas de vecteur de Killing, dans le second ceux-ci sont à dérivée covariante nulle. Le cas où $A_{ij}=\rho^2 g_{ij}$ est spécialement étudié, en particulier pour $\rho^2=1+c^2R^2$ (c=const.). Des résultats symétriques sont donnés en ce qui concerne les formes harmoniques.

A. Lichnerowicz (Paris).

Libermann, Paulette. Sur les connexions hermitiennes. C. R. Acad. Sci. Paris **239**, 1579–1581 (1954).

This note continues previous work [e.g., Ann. Mat. Pura Appl. (4) **36**, 27–120 (1954); MR **16**, 520]. A hermitian connection, introduced by Ehresmann and Libermann [C. R. Acad. Sci. Paris **232**, 1281–1283 (1951); MR **12**, 749], in an almost complex and almost hermitian manifold, is imbedded in a 2-parameter family $C_{a,b}$ of connections. These connections are investigated from the point of view of conformal behavior (multiplication of the metric by a positive differentiable function), n-isotropy, existence of almost Kähler structures, and related concepts. For $C_{1/2,b}$ the curvature equals the conformal curvature; if the structure is conformally integrable (i.e. locally there exist conformally equivalent integrable structures), then the restricted homogeneous holonomy group is trivial. *H. Samelson.*

Ehresmann, Charles. Extension du calcul des jets aux jets non holonomes. C. R. Acad. Sci. Paris **239**, 1762–1764 (1954).

Ehresmann, Charles. Applications de la notion de jet non holonome. C. R. Acad. Sci. Paris **240**, 397–399 (1955).

This continues previous work [e.g., same C. R. **234**, 1424–1425 (1952); MR **13**, 870]. A non-holonomic jet from the manifold V_n (cf. class $\geqq r$) to V_m is the jet associated with a cross-section over V_n, of class l, of the manifold $J^k(V_n, V_m)$ of k-jets from V_n to V_m. A number of related concepts are introduced; e.g.: general differential system (=manifold extracted from $J^k(V_n, V_m)$), non-holonomic prolongement of such a system; natural composition of non-holonomic jets, semi-holonomic jets and velocities (=jet from Euclidean space R^n to V_n); the invertible ones among the latter form a principal bundle over V_n; any associated bundle is called perfect prolongement. Second note: Semi-holonomic jets of order r from R^n to R^m $(0\to 0)$ can be represented as systems of tensors $(a_j{}^i, a^i{}_{j_1j_2}, \cdots, a^i{}_{j_1\cdots j_r})$, $i=1, \cdots, m$, $j_k=1, \cdots, n$; symmetry in the j_k makes the jet holonomic. A composition law $E\times E'\to E$ induces a non-holonomic prolongement of order r, pairing the non-holonomic jets from V_n to E and E' to those from V_n to E. For any subgroupoid ϕ of the groupoid of invertible semi-holonomic jets the notion of prolongement of V_n of type ϕ is defined (a manifold with a map into V_n, with ϕ as groupoid of operators), and a transitivity theorem for such prolongements is stated. *H. Samelson* (Ann Arbor, Mich.).

Apte, Madhumalati. Sur certaines classes caractéristiques des variétés Kähleriennes compactes. C. R. Acad. Sci. Paris **240**, 149–151 (1955).

Let $-\Phi/2\pi$ denote the Chern class of degree 2 of a compact Kähler variety V_{2n}. Then, after decomposing Φ into $\Phi=\alpha\beta+H_0+\lambda F$, where $H_0+\lambda F$ is harmonic, $\Delta H_0=0$, F=fundamental form of V_{2n}, and after applying the operators M_k of Lichnerowicz [Colloque de géométrie différentielle, Louvain, 1951, pp. 99–122; MR **13**, 688], the author obtains the formula:

$$(H_0, H_0)=\int R_{ij}R^{ij}\,dv-\frac{1}{2}\int R^2\,dv+\frac{n-1}{2nV}\left[\int R\,dv\right]^2.$$

From this she is able to conclude: A necessary and sufficient condition for the Chern class of degree 2 of V_{2n} to be cohomologous to an effective class is that $\int R dv=0$. In order that it be proportional to the fundamental class it is necessary and sufficient that the expression on the left vanish. A further result is that if the Chern class of degree 4 of an Einstein-Kähler variety is zero, the variety is necessarily Euclidean. *W. M. Boothby* (Evanston, Ill.).

Hermann, Robert. Sur les isométries infinitésimales et le groupe d'holonomie d'un espace de Riemann. C. R. Acad. Sci. Paris **239**, 1178–1180 (1954).

Hermann, Robert. Sur les automorphismes infinitésimaux d'une G-structure. C. R. Acad. Sci. Paris **239**, 1760–1761 (1954).

Let V_n be a Riemannian space with a G-structure and a G-connection ($G \subset O(n)$) in the sense of Chern, then the Riemannian connection gives rise to a holonomy group H with Lie algebra $\tilde{H}$ (a tilde denotes the Lie algebra of a group), and $\tilde{H} \subset \tilde{G}$. The author investigates when every one-parameter group of isometries of the Riemannian structure is also a group of isometries of the G-structure. The results are formulated in a few theorems, all making this same assertion under different assumptions. For Th. II he assumes that the G-structure is without torsion; that $\tilde{G} \supset \tilde{H}$ + the centralizer of $\tilde{H}$ in $\tilde{O}(n)$, and that $\tilde{H}$ is either semi-simple or irreducible; for Th. IV V_n should be compact and connected, without almost-complex structure, $G \subset L_n$ should be reducible and unimodular, and H irreducible; and for Th. V V_n is compact, the G-structure is without torsion, $G \subset SO(n)$, and the second Betti number vanishes. A corollary of Th. V states that the holonomy group of every homogeneous Riemannian metric on the sphere S^{2n} is $SO(2n)$.

Theorem III is a partial generalization of a theorem of Chern, and states that if V_n is compact, symmetric, and satisfies the conditions of Th. II, and if B^p denotes the space of p-vector fields such that $B^p(x)$ at each $x \in V_n$ is a subspace invariant under G, then every mapping $j_x: B^p(x) \to B^q(x)$ which is differentiable in x and commutes with the operations of G, sends the harmonic p-forms into harmonic q-forms.

The first paper contains a few errors which are corrected in the second. *A. Nijenhuis* (Princeton, N. J.).

Yano, K., and Davies, E. T. Contact tensor calculus. Ann. Mat. Pura Appl. (4) **37**, 1–36 (1954).

A set of N linearly independent contravariant vector fields B_i^A and C_r^A ($i=1, \cdots, n$; $r=n+1, \cdots, N$) in an X_n defines two non-holonomic subspaces $X_N{}^n$ and $X_N{}^{N-n}$ with B_i^A and C_r^A as basic system. The basic system and its reciprocal system (B_A^i, C_A^r) are specialized such that $B_j^A = (\delta_j^i, B_j^r)$ and $C_A^r = (C_j^r, \delta_s^r)$. A contact tensor calculus is developed by considering the group of transformations in X_N ($N=2n$) to be the group of homogeneous contact transformations and applying the theory of non-holonomic subspaces. Then the B_j^r and B_r^i are the first and second contact frame introduced by Eisenhart and Knebelman [Ann. of Math. (2) **37**, 747–765 (1936)] and Y. Mutô [Proc. Phys.-Math. Soc. Japan (3) **20**, 451–457 (1938)]. By introducing a metric in V_{2n} a generalization is obtained of geometries based on a covariant vector of support (C-space). It is shown how a C-space is transformed into a Finsler space. A discussion is given of the possibility of deriving a C-space (and consequently a Finsler space) from a Riemannian space by a homogeneous contact transformation. *J. Haantjes* (Leiden).

Takano, Kazuo. Contact transformations and generalized metric spaces. Tensor (N.S.) **4**, 51–66 (1954).

This paper is a contribution to the related studies of non-holonomic spaces, the invariant theory of contact transformations, and generalized metric spaces which have been treated in various papers listed by the author. The topics treated here are very closely related to some recent work by Yano and the reviewer [see the paper reviewed above] which was not available to the author at the time of writing. In the first part of this paper the author obtains Finsler spaces from metric spaces admitting homogeneous contact transformations, relating his work to that of Eisenhart [Ann. of Math. (2) **49**, 227–254 (1948); MR **9**, 380]. In the second part he connects up with Doyle [ibid. **42**, 698–722 (1941); MR **3**, 19] in his use of non-holonomic spaces.

E. T. Davies (Southampton).

Sasayama, Hiroyoshi. On the extended affine connection parameters of fractional order in the space with torsion. Tensor (N.S.) **3**, 144–166 (1954).

The author has introduced generalizations of the notion of extensor in two recent publications [Tensor (N.S.) **3**, 53–64, 101–107 (1953); MR **15**, 255, 900]. In this paper the author extends recent of work of Y. Katsurada [J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 17–28 (1951); MR **15**, 986] on ordinary extensors in a space with an affine connection without torsion to the case of extensors of fractional grade and to spaces with torsion. The author defines excovariant derivatives of extensors of the generalized types which he has introduced. By considering the permutability of two such excovariant derivatives, he arrives at curvature extensors, for which he establishes extensors of the usual identities satisfied by the Riemann-Christoffel tensors.

E. T. Davies (Southampton).

Craig, Homer V. On certain linear extensor equations. Tensor (N.S.) **4**, 40–50 (1954).

The author defines primary extensors of a function f of a set of variables x and of their derivatives up to an order m with respect to one or more parameters. He considers in particular the primary extensors associated with the key functions of certain problems of mathematical physics and of the calculus of variations, and studies in detail the structure of the linear equations which may be formulated from these primary extensors. He points out that the Euler equations for the first-order problems of the calculus of variations in parametric form of the simple and isoperimetric types can be expressed as linear equations of the type envisaged, so that his method would be applicable to any of the problems of mathematical physics which may be formulated as calculus-of-variations problems. The author illustrates his method by considering a number of one-parameter problems such as the isoperimetric problem of the calculus of variations and the Lagrangian equations of motion. He passes on to consider multiple-parameter problems from the same point of view. *E. T. Davies.*

Deicke, Arno. Finsler spaces as non-holonomic subspaces of Riemannian spaces. J. London Math. Soc. **30**, 53–58 (1955).

Let F_n be a Finsler space and L_{2n-1} be the associated space of line-elements. The author proves that it is possible to introduce into L_{2n-1} a Riemannian metric and an affine connection with torsion which will induce in a suitably chosen non-holonomic manifold $X^n{}_{2n-1}$ the metric and connection of F_n. It is assumed that the metric and torsion of L_{2n-1} depend only on the local length and angular metric of F_n. The results are valid for the whole of L_{2n-1}.

C. B. Allendoerfer (Seattle, Wash.).

Ohkubo, Takeo. On relations among various connections in Finslerian space. Kumamoto J. Sci. Ser. A. 1, no. 3, 1–6 (1954).

In this paper the different connection coefficients of covariant derivatives in Finsler spaces as defined by Cartan, Berwald and the reviewer are compared. Possible relations between arbitrary metric connections (i.e. connections for which the covariant derivative of the metric tensor vanishes) are studied. Using criteria similar to those of the reviewer a new metric connection is introduced whose coefficients are not evaluated explicitly, since their expressions involve arbitrary tensors. *H. Rund* (Toronto, Ont.).

Longo, Carmelo. Sui complessi lineari di piani. Ann. Mat. Pura Appl. (4) **37**, 61–138 (1954).

The classification of trivectors in E_n is accomplished for $n \leqq 8$. Since a scalar factor is not important, the trivectors may be represented by linear complexes of planes in a projective $(n+1)$-dimensional space. This gives a possibility of performing the classification by using geometric means only and of finding constructions and geometric particulars for all possible cases. This is done in this paper after an introduction giving full account of the results already found by others. Figures are given for $n=6$ (2 cases), $n=7$ (5 cases) and $n=8$ (13 cases). For $n=8$ classifications were given by G. B. Gurevič [Mat. Sb. N.S. **27(69)**, 103–116 (1950); MR **14**, 586] and by L. C. Hutchinson in his dissertation [Mass. Inst. Tech., 1940] which seems not to be available. In a note presented in 1948 [Bull. Amer. Math. Soc. **55**, 277 (1949)], Hutchinson calls the form of a trivector canonical if it is written as the sum of alternating products of 3 of the n basis vectors in such a way that two products never have more than one factor in common. It is never proved that this is possible for every value of n, but in this note Hutchinson gave canonical forms for each of the 13 types for $n=8$. Since the 13 types of Longo are not in canonical form, the two results can not be compared immediately. *J. A. Schouten* (Epe).

NUMERICAL AND GRAPHICAL METHODS

***Mikeladze, Š. E. Čislennye metody matematičeskogo analiza. [Numerical methods of mathematical analysis.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 527 pp. 21 rubles.

Except for a few classical texts on finite differences the field of numerical analysis has lain fallow for a century or more until about the last decade. With awakened interest in computational methods new texts have been appearing with increasing frequency, mainly in English, but also in German, French, and Italian. Now Mikeladze, productive writer in numerical methods, has come up with one of the largest (527 p.) treatises yet to appear. It deals with so many current topics that it is perhaps easier to say what is not covered than to list its contents. Among numerical methods of current interest the main ones omitted deal with linear equations and matrices, and with numerical solution of differential equations. Perhaps the existence of Faddeeva's book [Computational methods of linear algebra, Gostehizdat, 1950; MR **13**, 872] accounts for the omission of the former and possibly Mikeladze, who is something of a specialist on differential equations, intends also to write a book on numerical solutions.

In the quality of paper and typography, in the careful setup of mathematical formulas, and in the scholarly character of the presentation, this book makes a good impression. One characteristic is interesting. In America, where calculating machinery, both desk type and large scale electronic, is becoming increasingly available, the trend in numerical procedures is away from the classical method of differences, a method devised, in part at least, to ease the drudgery of many long multiplications. However Mikeladze adheres to the classical practice, as we note from the titles of the early chapters: I) Finite differences; II) Finite sums; III) Divided differences; IV) Reciprocal differences.

In Ch. V he turns to the basic problem of best approximations, develops the theorems of Bernstein and introduces the Čebyšev polynomials. Ch. VI is devoted to polynomial interpolation, starting from the determinant form, and presenting Newton's formula in terms of divided differences, first for unequal, then for equal intervals, and finally for the complex domain. The formulas of Lagrange, Gauss, Simpson, Bessel, Everett et al., follow. The convergence of the interpolating process and the magnitude of the error are considered. Shepard's rules for the formation of interpolating polynomials and the iteration processes of Aitken and Neville are likewise explained. The author then treats interpolation by rational functions, using continued fractions and reciprocal differences; he next takes up trigonometric interpolation; and finally polynomial interpolation with confluent points, where derivatives as well as ordinates of the function are employed.

Ch. VII, entitled quadratic approximation, deals with the problem of approximating a given function in terms of some basic set of functions by the method of least squares. This introduces the idea of orthogonal functions, particularly polynomials orthogonal over a discrete set of points. Ch. VIII goes on to consider Fourier series and orthogonal polynomials, orthogonal over a continuous interval. Ch. IX treats empirical formulas, Ch. X construction of numerical tables, Ch. XI inverse interpolation, and XII deals with numerical differentiation. A long Ch. XIII (115 pages) is devoted to numerical integration. It includes all the classical formulas of the Newton-Cotes closed type and open type, as well as a list of formulas for integration over m steps using k ordinates for a considerable number of values of m and k. Magnitude of error and comparison of accuracy of many formulas is dealt with. The author also treats the quadrature formulas of Gauss, Čebyšev, and Markov, the investigations of Bernstein and Steklov, quadrature formulas in terms of differences, numerical approximation of improper integrals, method of undetermined coefficients, multiple integrals, and inequalities for estimating definite integrals.

Ch. XIV is devoted to Euler's summation formula, XV to summation by differences, XVI to multiple summation, XVII to interpolation with several variables, XVIII to cubature, and XIX to symbolic calculus. The book concludes with an 8-page bibliography. *W. E. Milne.*

***Zeuli, Tino. Introduzione ai calcoli numerici e grafici.** Editrice Gheroni, Torino, 1950. 419 pp.

This book is a paper-covered lithographed set of typewritten notes based on the author's course on numerical and graphical calculus at the University of Torino. The topics, which are treated in a simple and clear style, include elementary operations, calculating instruments, numerical

tables, approximate calculations, the slide rule, calculations of polynomials and of series. Then comes a discussion of interpolation, including the most notable classical interpolating formulas, and inverse interpolation. There follows the solution of equations by the regula falsi, Newton's method, Graeff's method, and iteration. Then we have numerical integration and differentiation with the classical formulas as well as various mechanical integrating devices. Then we have graphical and numerical solution of differential equations, a treatment of equations in several unknowns, and finally a chapter on empirical formulas.

W. E. Milne (Corvallis, Ore.).

***Tables of sines and cosines for radian arguments.** National Bureau of Standards Applied Mathematics Series, no. 43. U. S. Government Printing Office, Washington, D. C., 1955. xiii+278 pp. $3.00.

Reissue of an earlier set of tables [New York, 1940; MR 2, 64] with correction of known errors and minor modifications and additions.

***Hayashi, Keiichi. Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen sowie der Funktionen e^x und e^{-x} mit den natürlichen Zahlen als Argument.** Neudruck. Walter de Gruyter & Co., Berlin, 1955. ii+182 pp. DM 12.00.

Miller, George A., and Ross, Patricia M. Tables of $n \log_2 n$ and $n \log_{10} n$ for n from 1 to 1000. Lincoln Laboratory, Massachusetts Institute of Technology, Tech. Rep. no. 60, 9 pp. (1954).

These two little tables give values having mostly seven significant digits of the functions mentioned in the title for $n=1(1)1000$. The first half of the table was prepared by hand. The work was then verified and completed on the Whirlwind. *D. H. Lehmer* (Berkeley, Calif.).

Steinhaus, H. Shuffled numbers. Zastos. Mat. 2, 34–45 (1954). (Polish. Russian and English summaries)

The author gives a second account of his table of shuffled numbers [Rozprawy Mat. 6 (1954); MR 15, 636] which is an alledgedly random permutation of the integers from 0 to 9999. He criticizes the genesis of the early tables of random numbers as not being purely mathematical. If the present table has been subjected to any of the standard tests for randomness the author fails to say so. One page of the table, containing 1600 decimal digits, is reproduced on p. 40.

D. H. Lehmer (Berkeley, Calif.).

***Engvall, Albert. A formula for the computation of Gauss's error integral.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 40–41 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution). (Swedish)

A variation on the familiar evaluation of $\Gamma(\frac{1}{2})$ by transformation to a double integral shows that $I=\int_0^\infty \exp(-x^2)\,dx$ satisfies $aI^2=(4\pi)^{-1}[a-I\exp(-a^2)]$ approximately. Comparison of the solution of this quadratic with exact values suggests a relative error of at most .01 for $0 \leqq a < \infty$.

John Todd (New York, N. Y.).

***Altenkirch, Edmund. Verzögerungsfunktion.** Verlag Technik, Berlin, 1952. 75 pp. (6 plates).

This is an elaborate (and sometimes obscure) account of the delay-function (Verzögerungsfunktion) $\phi(x)$ defined by: $\phi(x)=0$, $-\infty<x\leqq -1$; $\phi(x)=1$, $-1<x\leqq 0$; $\phi(x)=\sum(-)^n[(x-n)\alpha]^{n+1}/(n+1)!$ for $x\geqq 0$, where the summation is over the range $0\leqq n<x$. It satisfies the difference-differential equation $\phi'(x)=-a\phi(x-1)$. If $0<\alpha<\epsilon^{-1}$, then as $n\to\infty$ we have $\phi\to A_0\exp(-bx)$ where

$$A_0=\sum_0^\infty[(n-1)\alpha]^n/n!$$

and b is the smaller of the two roots of $\alpha=b\exp(-b)$. Finally $\phi(x)\to a\sin\beta(x+\gamma)$ as $x\to\infty$ only if $\alpha=\frac{1}{2}\pi$. This function is said to be of importance in the study of many types of control mechanisms.

There are many graphs and tables of the functions themselves together with various related and auxiliary functions. In particular, there are tables of the coefficients $(x-r)^n/n!$, mostly at intervals of 1 in x, and for n up to 20. These have been spot checked with the tables of $x^n/n!$ in Nat. Bur. Standards, Appl. Math. Ser. no. 37 (1954) [MR 16, 402]. Two major discrepancies were noted (at $x=10.4$, $r=9$, $n=10$ and $x=8.6$, $r=7$, $n=8$) in addition to many involving the last or last two places. *John Todd.*

Chandrasekhar, S., and Elbert, Donna D. The illumination and polarization of the sunlit sky on Rayleigh scattering. Trans. Amer. Philos. Soc. (N.S.) 44, 643–728 (1954).

The physical problem indicated in the title, and the mathematical solution of that problem, have been presented in Chapter X of Chandrasekhar's "Radiative transfer" [Oxford, 1950; MR 13, 136]. The solution depends on two functions $X(\mu)$ and $Y(\mu)$ which satisfy the system of integral equations

$$X(\mu)=1+\int_0^1\frac{\mu}{\mu+\mu'}\Psi(\mu')[X(\mu)X(\mu')-Y(\mu)Y(\mu')]\,d\mu'$$

$$Y(\mu)=e^{-\tau/\mu}+\int_0^1\frac{\mu}{\mu-\mu'}\Psi(\mu')[Y(\mu)X(\mu')-X(\mu)Y(\mu')]\,d\mu'.$$

There are four functions Ψ occurring in this connection,

$$\Psi_I(\mu)=\tfrac{3}{4}(1-\mu^2),\quad \Psi_r(\mu)=\tfrac{3}{8}(1-\mu^2)$$
$$\Psi^{(1)}(\mu)=\tfrac{3}{8}(1-\mu^2)(1+2\mu^2),\quad \Psi^{(2)}(\mu)=\tfrac{3}{16}(1+\mu^2)^2,$$

and the corresponding X and Y carry the same index as Ψ. In the case of Ψ_I, the solution of the integral equations is not unique, and "standard solutions" are determined by the conditions

$$\int_0^1 X_I(\mu)\Psi_I(\mu)d\mu=1,\quad \int_0^1 Y_I(\mu)\Psi_I(\mu)d\mu=0.$$

[For some earlier tabulations see Chandrasekhar, Elbert, and Franklin, Astrophys. J. 115, 244–268 (1952); Chandrasekhar and Elbert, ibid. 115, 269–278 (1952); MR 13, 871; and Tables relating to Rayleigh scattering of light in the atmosphere, Univ. of California, Dept. of Meteorol., 1952].

Table I of the present memoir gives 5D values of X_I, Y_I, X_r, Y_r, $X^{(1)}$, $Y^{(1)}$, $X^{(2)}$, $Y^{(2)}$ for $\mu=0(.01)1$, $\tau=.05(.05).5, 1$. Initial values were computed by the second-named author from the corrected second approximation given in Radiative Transfer, §60. Starting with these, the integral equations were solved by iteration at the Watson Scientific Computing Laboratory.

Tables II to XII contain numerical material of interest for the solution of the physical problem. These tables form the bulk of the work, but they do not lend themselves to a description in general terms. *A. Erdélyi.*

Hartree, D. R. Approximate wave functions, with exchange for Mn^{+2}. Proc. Cambridge Philos. Soc. **51**, 126–130 (1955).

Formulae and a table are given for writing down Fock's equations for the radial wave functions for the lowest term of a configuration consisting of an incomplete $(3d)$ group in addition to complete groups. Preliminary numerical results are given for the ^6S term of the $(3d)^5$ configuration of the Mn^{+2} ion. (Author's summary.) *A. Erdélyi.*

***van Wijngaarden, A. Erreurs d'arrondiment dans les calculs systématiques.** Les machines à calculer et la pensée humaine, pp. 285–293. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

L'auteur étudie d'abord le bruit, c'est à dire l'erreur d'aspect aléatoire du à l'arrondi. On aurait tendance à croire que sa répartition est sensiblement normale. Il n'en est pas toujours ainsi. L'auteur étudie ensuite les erreurs dues à l'arrondi des paramètres, avec applications aux fonctions exponentielle et trigonométriques. L'étude de l'équation du second degré, de la détermination de arc cos x de l'intégration de $y''=y$ fournissent des exemples où la répartition des erreurs n'est pas celle que l'on attendrait à première vue. *J. Kuntzmann* (Grenoble).

Trimble, George R., Jr. Techniques for computing linear sums. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Tech. Note No. **897**, 11 pp. (1954).

Four methods are discussed for obtaining the smoothed value of $N=2n+1+p$ observed values ($p=0$ or 1): linear combinations (1) of observed values, (2) of central-differences, (3) of power moments and (4) of factorial moments. Abel's partial summation formula is introduced to reduce the size of the numbers used in (1). Recursion relations are presented to simplify computing a sequence of smoothed values by use of (3) or (4). Recursion relations are also given for general types of moments. *R. L. Anderson.*

Trimble, George R., Jr. Computing derivatives of least square polynomials using differences. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Tech. Note No. **898**, 7 pp. (1954).

Given points (X_i, T_i), $i=1, 2, \cdots, n$, where X_i is a position coordinate and T_i is the time ($\Delta T_i=h$, a constant). It is desired to fit a quadratic function to the data:

$$\hat{X}_i=x_i=a_0+a_1 t_i+a_2 t_i^2,$$

where $t_i=(T_i-\bar{T})/h=i-\frac{1}{2}(n+1)$. The final objective is to evaluate the velocity and acceleration components at the midpoint of the interval; these are simply multiples of a_1 and a_2. Formulas are developed to compute a_1 and a_2 by use of power moments of order 0, 1, 2 and 3 of the first differences of the X's. A table of the coefficients of these power moments is given for $N=2(2)24$.

R. L. Anderson (Raleigh, N. C.).

Trimble, George R., Jr. Some relations involving coefficients used in certain types of linear sums. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Tech. Note No. **899**, 13 pp. (1954).

We consider the linear combination of $N=2n+p+1$ observed values

$$U=\sum_{i=-n}^{n+p} c_i u_i,$$

where $p=0$ or 2 for odd N and $p=1$ for even N and the c_i are symmetric. Four cases are considered: I. N even and $c_{i+1}=c_{-i}$; II. N odd ($p=0$) and $c_i=c_{-i}$ (c_0 not paired); III. N even and $c_{i+1}=-c_{-i}$; IV. N odd ($p=2$) and $c_{i+2}=-c_{-i}$ (c_1 not paired). U can also be written in terms of central differences,

$$U=\sum_{j=0}^{n} d_j(\Delta^{2j}u_{-j}+\Delta^{2j}u_{-j+1}).$$

Formulas are derived to solve for the d's in terms of the c's and vice versa for each of the four cases. Hence, if one set of constants is derived from a set of data, while it is easier to use the other form of U, the transformation is readily made. *R. L. Anderson* (Raleigh, N. C.).

Fritz, W. Barkley. A comparison of machine methods for evaluating certain mathematical functions. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. **774**, 13 pp. (1954).

Elementary considerations of approximate evaluation on automatic computers of exponential and direct and inverse trigonometrical functions by polynomials (of fixed degree), by polynomials (of varying degree determined by actual argument) and by continued fractions. *John Todd.*

***Scherberg, M. G., and Riorden, J. F. Analogue calculation of polynomials and their zeros.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 118–120. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Bodewig, K. E. Die Inversion geodätischer Matrizen.** Den Haag, 1954. 81 pp. (mimeographed) (may be ordered from T.N.O. (Toegepast Natuurwetenschappelijk Onderzoek), the Hague, Holland).

In triangulation the direction measurements are adjusted by the method of least squares. The coefficients of the normal equations form the "geodetic matrices" of the title; the problem of adjustment is essentially to invert these matrices. In a geodetic matrix G a large submatrix A (corresponding to the "angle equations") is mostly zeros and depends only on the topology of the network of stations and observed directions. When the directions are given equal weights, $\frac{1}{3}A$ has 3's on the main diagonal and ± 1's in a few positions off the diagonal.

H. Boltz proposed first obtaining A^{-1} (which can be done before the survey), and then using it to get G^{-1} by partitioning G. Papers by K. Friedrich and W. Jenne have described how to obtain A^{-1} for various nets of simple structure. [For background see Boltz, Veröff. Preuss. Geodät. Inst. Potsdam (N.F.) no. **90** (1923); Jenné, ibid. no. **107** (1937); Friedrich and Jenne, Deutsche Akad. Wiss. Berlin, Veröff. Geodät. Inst. Potsdam no. **5** (1951); MR **13**, 387].

Bodewig shows how to calculate A^{-1} explicitly without round-off error for both open and closed simple chains of k triangles, and gives simple tables specifying A^{-1} for all $k\leq 15$. There is no reference to earlier work on the inversion of these matrices [e.g., D. E. Rutherford, Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 232–241 (1952); MR **15**, 495].

When the network of triangles has a more intricate coupling, a partitioning method is suggested for getting A^{-1}, where one starts with the matrix for the longest simple chain one can find, and then borders it with one, two, or three columns. There are several numerical examples explicitly solved.

In one section the author proposes the exact numerical inversion of a matrix B of integers by carrying out the textbook reduction to Smith normal form. One essentially reduces B to the identity by row elimination with exact integers, and then does the same thing to the unit matrix to get B^{-1}. [Reviewer's comment: It would appear from the critique of this type of method given by Rosser [J. Res. Nat. Bur. Standards **49**, 349–358 (1952); MR **14**, 1128] that for general matrices the intermediate integers commonly grow intolerably fast in the early stages. For geodetic matrices the objection does not hold, and the method could be very useful.]

One section deals with correcting the inverse of a geodetic matrix after one element is altered. In the last section the author discusses inverting G by a common iterative process, assuming that A^{-1} is known, but he is unable to settle the question of convergence.

Throughout the book the author is interested in avoiding round-off error by computing the exact adjoint matrices, and discusses controls for calculations. *G. E. Forsythe.*

*Forsythe, A. I., and Forsythe, G. E. **Punched-card experiments with accelerated gradient methods for linear equations.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 55–69. National Bureau of Standards Applied Mathematics Series No. 39, U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

The present paper contains a description of a number of interesting and instructive experiments on the use of gradient methods. It is pointed out that for an effective use of gradients, it is normally necessary to devise methods for accelerating iterations based on the gradient method. Essentially two types of acceleration are studied. The first type is that obtained by overrelaxing and underrelaxing. The second type is that obtained by interspersing an acceleration of the type $x_{k+1}=\alpha x_k+(1-\alpha)x_{k-2}$ amongst iterations of the form $x_{k+1}=x_k+\alpha g_k$, where g_k is the gradient of the error function to be minimized. A theoretical discussion of these devices is given. Although these experiments were carried out on a slower type machine, it is undoubtedly true that accelerations are useful on high-speed machines. [In Table 4 on p. 66, bottom line, 2d column, read 26.3 instead of 33.0.] *M. R. Hestenes* (Oslo).

*Wheeler, D. J., and Nash, J. P. **Digital computer methods for solving linear algebraic equations and finding eigenvalues and eigenvectors.** Digital and analog computers and computing methods, pp. 21–35. The American Society of Mechanical Engineers, New York, N. Y., 1953. $3.00.

This is a general discussion of the nature and capabilities of binary electronic digital computers. As illustrative examples the authors discuss in some detail two matrix algebra codes for the University of Illinois Digital Computer (ILLIAC). The first code solves a system of n simultaneous linear algebraic equations ($n\leqq 39$) by Gaussian elimination in less than 3 minutes. The second obtains all eigenvalues ($n\leqq 40$) or all eigenvalues and eigenvectors ($n\leqq 23$) of a real symmetric matrix of order n by a variant of Jacobi's process in about $n^3/25$ seconds. A flow chart and typical orders are included.

The authors advocate avoiding "floating-point" arithmetic on the grounds that it is about 30 times slower than use of a fixed point. [Reviewer's note: Fast engineering development seems already to have replaced the factor 30 by something approaching unity on the newest machines.] *G. E. Forsythe* (Los Angeles, Calif.).

Lotkin, Mark. The propagation of error in numerical integrations. Proc. Amer. Math. Soc. **5**, 869–887 (1954).

Reviewed earlier in report form [Ballistic Res. Lab., Aberdeen Proving Ground, Rep. no. **875** (1953); MR **16**, 405].

Coulmy, Geneviève. Méthode de calcul des intégrales de Lommel généralisées. Ann. Télécommun. **9**, 305–312 (1954).

In order to compute the integrals

$$I_{p,q}=\int_0^b x^\gamma J_p(\alpha x)J_q(\beta x)dx$$

numerically, the author uses recurrence formulas of Bessel functions to prove the expansion

$$I_{p,q}=\frac{2}{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}G_{mn}(p,q,\gamma)M_{p+m,q+n}(\alpha,\beta,\gamma,a,b),$$

where

$$M_{p+m,q+n}(\alpha,\beta,\gamma,a,b)=b^\gamma J_{p+m}(\alpha b)J_{q+n}(\beta b)-a^\gamma J_{p+m}(\alpha a)J_{q+n}(\beta a)$$

and the G_{mn} are numerical coefficients which are independent of a, b, α, β, and for which explicit expressions are given in the paper. In the case of integer values of γ, the computation of the G_{mn} is much simplified. Several other methods of computation of $I_{p,q}$ are indicated briefly. *A. Erdélyi.*

Czerny, M. Zur Integration des Planckschen Strahlungsgesetzes. Z. Physik **139**, 302–308 (1954).

The convenience of tabulating a non-dimensional form of the Planck radiation integral is pointed out. It is suggested that $B(v)=15\pi^{-4}\int_0^v[t^5(\exp t^{-1}-1)]^{-1}dt$, and its first two derivatives be tabulated in place of the

$$R=\int_0^\lambda c_1\lambda^{-5}(\exp(c_2/\lambda T)-1)^{-1}d\lambda,$$

where c_1, c_2 depend on the velocity of light, Planck's constant and Boltzmann's constant. The function R was tabulated by A. N. Lowan and G. Blanch [J. Opt. Soc. Amer. **30**, 70–81 (1940); reprinted in Miscellaneous physical tables, Nat. Bur. Standards, 1941; MR **1**, 252; **4**, 282]. A table of (a multiple of) $B'(v)$ was given by F. Emde [Tafeln elementarer Funktionen, Teubner, Leipzig, 1941, pp. 117–119; MR **3**, 152]. *John Todd* (New York, N. Y.).

Radok, J. R. M. The solution of eigenvalue problems of integral equations by power series. Quart. Appl. Math. **12**, 413–417 (1955).

For eigenvalue problems of the type

$$\varphi(x)=\lambda\int_0^1 L(x,y)\varphi(y)\,dy$$

with $L(x,y)=P(x,y)+|x-y|\cdot Q(x,y)$; P and Q symmetrical polynomials in x and y with real coefficients, the author proposes to set up the eigenfunction as a power series $\varphi(x)=\sum_{k=0}^{\infty}a_kx^k$. This leads to conditions on λ and on the coefficients a_k. The author presents all the formulas so as to apply this method and he considers as an example the case $P=0$, $Q=1$, for which the infinite power series can be summed up so as to result in hyperbolic functions.

H. Bückner (Schenectady, N. Y.).

Bukovics, Erich. Beiträge zur numerischen Integration. III. Nachträge. Monatsh. Math. **58**, 258–265 (1954).

In two earlier papers of this series [Monatsh. Math. **57**, 217–245 (1953), 333–350 (1954); MR **15**, 561] the author gave error estimates for the Runge-Kutta process and the Blaess process of solving differential equations numerically. In the estimates as given in the two earlier papers there appear three quantities M_2, M_3, M_4, upper bounds, respectively, for the second, third, and fourth total derivatives of $f(x, y)$, where $y'=f(x, y)$. The present paper extends the hypotheses of the first two papers so as to make possible a determination of M_3, M_3, and M_4, and hence provide more convenient expressions for the error estimates in the case of the Runge-Kutta and the Blaess processes.

W. E. Milne (Corvallis, Ore.).

Babkin, B. N. Approximate integration of systems of ordinary differential equations of the first order by the method of S. A. Čaplygin. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 477–484 (1954). (Russian)

Čaplygin's method of comparison functions for the approximate solution of a differential equation applies in the first instance to a single equation of the first order. In order to extend it to two equations

$$\frac{dy}{dx}=f(x, y, z), \quad \frac{dz}{dx}=\varphi(x, y, z)$$

with $y(x_0)=y_0$, $z(x_0)=z_0$ the author presents the fundamental theorem: If by any means we find four functions $u(x)$, $v(x)$, $s(x)$ and $t(x)$ belonging to class $c_1[x_0, x_1]$ and satisfying the following two conditions: (a) the space curves $y=u(x)$, $z=s(x)$ and $y=v(x)$, $z=t(x)$ pass through the point (x_0, y_0, z_0) and for $x_0 \leqq x \leqq x_1$ lie entirely in a given bounded region, and (b)

$$\frac{dy}{dx}-f(x, u, \tau)<0, \quad \frac{dv}{dx}-f(x, v, \tau)>0$$

for any $\tau(x)$ included between $s(x)$ and $t(x)$ (including s and t),

$$\frac{ds}{dx}-\varphi(x, \sigma, s)<0, \quad \frac{dt}{dx}-\varphi(x, \sigma, t)>0$$

for any $\sigma(x)$ included between $u(x)$ and $v(x)$ (including u and v) for all x in the segment $[x_0, x_1]$, then the inequalities

$$u(x)<y(x)<v(x), \quad s(x)<z(x)<t(x)$$

hold for $x_0<x<x_1$, where $y(x)$ and $z(x)$ are solutions of (1) through (x_0, y_0, z_0). *W. E. Milne* (Corvallis, Ore.).

Sconzo, P. Formule di estrapolazione per l'integrazione numerica delle equazioni differenziali ordinarie. Boll. Un. Mat. Ital. (3) **9**, 391–399 (1954).

The author obtains from Lagrange's formula a set of extrapolation formulas giving, e.g., y_n in terms of y_0, y_1, $\cdots$, k, where $n>k$. With the aid of these and Euler's summation formula he derives a set of relations for predicting y_n in terms of earlier values of y and y'. These include the standard open-type Newton-Cotes quadrature formulas and are used to solve equations of the type $y'=F(x, y)$. Similar relations are obtained for predicting y_{n+1}, useful when one needs to cut the interval in half. The method is extended to the case of differential equations of the form $y''=F(x, y)$. *W. E. Milne* (Corvallis, Ore.).

Adachi, Ryuzo. On the numerical solution of the second order differential equation under some conditions. Kumamoto J. Sci. Ser. A. **1**, no. 3, 14–33 (1954).

The author considers a second-order differential equation $y''=f(x, y, y')$ and seeks a solution $y(x)$ having the property that at the intersections of $y=y(x)$ with two given curves $u_1(x, y)=0$, $u_2(x, y)=0$ at the points (x_1, y_1), (x_2, y_2), respectively, the given relations $g_1(x_1, y_1, P_1)=0$, $g_2(x_2, y_2, P_2)=0$ are to be satisfied, P being a value of y'. In general terms his procedure may be described as follows: starting from some assumed function satisfying the intersection conditions, he uses successive substitutions, carrying out the integrations numerically, to obtain improved approximations. As the reader will see, this requires adjustable parameters which have to be calculated to satisfy the conditions at the intersections. There is no assurance that the process will converge or that if a limit is reached it will represent a solution. However, if the approximations approach a limit within the degree of accuracy required, we may take the initial values at some point as given by this supposed solution and by straight-forward numerical integration calculate a solution of the differential equation with these initial values. If this computed solution satisfies the intersection conditions with sufficient accuracy we may accept it as a solution. *W. E. Milne* (Corvallis, Ore.).

Adachi, Ryuzo. On the numerical solution of the simultaneous differential equations under some conditions. I. Kumamoto J. Sci. Ser. A. **1**, no. 4, 28–30 (1954).

This is an extension of the preceding paper by Adachi to the case of three simultaneous equations of the type $du_i/dx=F_i(x, u_1, u_2, u_3)$, where intersection conditions are applied to each of the plane curves $u=u_i(x)$. Again the solution is obtained by successive approximations.

W. E. Milne (Corvallis, Ore.).

Mikeladze, Š. E. Numerical solution of boundary problems for nonlinear ordinary differential equations. Soobšč. Akad. Nauk Gruzin. SSR **14**, 133–137 (1953). (Russian)

The differential equation $y^{(n)}(x)=f(x, y, y', \cdots, y^{(n-1)})$ is replaced by the system

$$y^{(k)}(x)=\sum_{\nu=0}^{n-k-1} x^\nu y^{(k+\nu)}(0)/\nu!$$

$$+[(n-k-1)!]^{-1}\int_0^x (x-t)^{n-k-1}y^{(n)}(t)\,dt,$$

where $k=0, 1, \cdots, n-1$. The author then proposes to replace the integrals of this system by numerical approximations, with x restricted to the discrete set of values $x=\sigma h$, σ an integer. The resulting algebraic system is solved for y and its derivatives at arguments of this set. In the example chosen, $y''=ay(1-0.5y'^2)$, this final solution is accomplished by iteration. *R. E. Gaskell* (Seattle, Wash.).

***Mikeladze, Sh. E. Numerical solution of boundary problems for nonlinear ordinary differential equations.** Morris D. Friedman, Russian Translation, Two Pine Street, West Concord, Mass., 1954. 6 pp. (mimeographed). $3.00.

Translation of the paper reviewed above.

van den Dungen, F. H. Le principe de Rayleigh dans le cas des oscillations amorties. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 1038–1045 (1954).

In an earlier paper [same Bull. (5) **38**, 695–704 (1952); MR **14**, 412] the author shows that one may obtain characteristic frequencies either (a) by applying Newton's regula falsi to the characteristic equation, or (b) by Rayleigh's principle using an arbitrary deformation of the system, and shows that starting from the same first guess λ_0, both methods give the same sequence of successive approximations. In the present paper these results are extended to the case of damped vibrations. This requires a generalization of Rayleigh's principle to make it applicable to non-conservative systems. As in the first paper the actual proof is carried out only in the special case of a vibrating beam simply supported at the ends. *W. E. Milne* (Corvallis, Ore.).

***Ku, Y. H. Acceleration plane method for nonlinear oscillations.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 129–153. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

The differential equation $\ddot{x}+f(\dot{x}, x)+f_1(x)=K$ is studied by means of trajectories in the (v, x), (g, x), and (g, v) planes $(g=\dot{x}, v=\dot{x})$. A step-by-step graphical procedure is given for the simultaneous construction of the trajectories. Illustrations with diagrams are given for the damped linear oscillator, van der Pol's equation, and the pendulum with nonlinear restoring force. The application of the technique to a third-order equation is discussed. *W. S. Loud.*

Chandrasekhar, S. On characteristic value problems in high order differential equations which arise in studies on hydrodynamic and hydromagnetic stability. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953. Amer. Math. Monthly **61**, no. 7, part II, 32–45 (1954).

This article is an exposition of a method developed and applied by the author in a series of twelve papers, partly published and partly in press, a list of which is given in the bibliography. Two methods are explained by means of which the eigenvalues of numerous problems of hydrodynamic stability can be determined with comparatively little computational effort. The differential equations are ordinary and linear and are characterized by the occurrence of products of powers of simple differential expressions, such as, for instance, d^2/dx^2-a^2. Making use of this structure the author constructs, for self-adjoint problems, equivalent variational problems which lend themselves well to simple numerical techniques. For non-selfadjoint problems the differential equation is split into a system of two equations, which are then solved by series in terms of suitable orthogonal functions. If the splitting is done skillfully, the rate of convergence will be considerably higher than if similar series had been computed for the original problem. The paper concludes with a list of nine problems to which these methods have been applied. *W. Wasow* (Rome).

***Rabinowitz, P. The use of sub-routines on SEAC for numerical integrations of differential equations and for Gaussian quadrature.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 88–89. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

Robinson, Lawrence Baylor. Effect of delayed fission neutrons on reactor kinetics. J. Appl. Phys. **26**, 52–56 (1955).

The point of this paper is not clear. With special assumptions one can describe by a system of ordinary, nonlinear differential equations the effect of delayed neutrons upon reactor kinetics. The author exhibits such a system. Then in terms of a quite general system he gives the Picard iteration (without so naming it); alludes to "other methods"; defines the "concept of stability"; and warns against relying solely upon the approximating linear system in testing for it.

A. S. Householder (Oak Ridge, Tenn.).

***van den Dungen, F. H. L'intégration numérique de l'équation des ondes.** Les machines à calculer et la pensée humaine, pp. 215–238. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

The author considers numerical solution of the problem

$$\sum_i \frac{\partial^2 u}{\partial x_i^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (-\infty<x<+\infty, t\geqq 0)$$

$$u(x_i, 0)=f(x_i), \quad \frac{\partial u(x_i, 0)}{\partial t}=\varphi(x_i) \quad (i=1, 2, \cdots, N),$$

especially for $N\geqq 2$. A grid is laid over the region of integration with $\Delta x_i=\Delta$, $\Delta t\leqq\Delta/c$. Approximating u, f, φ by polynomials of second (or fourth) degree, he develops recurrence relations for passing from $u(x_i, t)$ to $u(x_i, t+\Delta t)$. The author considers error growth during the numerical solution, and proposes a new method for its control; this involves solving $(N+1)$ problems of the above sort, with the dependent variables being the $(N+1)$ partial derivatives of u. The author shows how more general problems, involving the general linear second-order hyperbolic equation (non-homogeneous, with constant coefficients) can be reduced to the problem above. *M. A. Hyman* (Pittsburgh, Pa.).

***Wegstein, J. H. A numerical solution of the helium wave equation with the SEAC.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 34–36. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Ladd, D. W., and Sheldon, J. W. The numerical solution of a partial differential equation on the IBM Type 701 electronic data processing machines.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 115–117. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

Yuškov, P. P. On improvement of the convergence of series obtained in a refined harmonic analysis. Inžen. Sb. **19**, 171–178 (1954). (Russian)

In two earlier papers [Inžen. Sb. **6**, 197–210 (1950); **10**, 213–222 (1951); MR **13**, 288, 874] the author proposed methods to obtain approximate values for the Fourier coefficients A_m and B_m of a periodic function from the coefficients α_m and β_m obtained by harmonic analysis. The scheme was based on representing the given function by segments of parabolas, continuously joined, but with discontinuous derivatives. Observing that discontinuities in the first derivatives adversely affect the convergence of the associated Fourier series, the author now shows that the discontinuities on the first derivative can be subtracted out and new formulas obtained to give better approximations for the

Fourier coefficients of the original function in terms of the coefficients α_m and β_m found by harmonic analysis.
W. E. Milne (Corvallis, Ore.).

de Finetti, B., Rodinò, G., e Kitz, N. "Symposium on automatic digital computation" (Nat. Phys. Lab., Teddington (Londra), 25–28 marzo 1953). Ricerca Sci. **23**, 1248–1259 (1953).

***Malavard, L. Le laboratoire de calcul expérimental analogique de l'Institut Blaise Pascal (Centre National de la Recherche Scientifique).** Journées de Mécanique des Fluides, Marseille 1952, pp. 95–104. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 296 (1951).

Michel, J. G. L. Errors of friction wheel integrators. J. Sci. Instr. **32**, 43–44 (1955).

Fritz, Norman L. Analog computer for nonlinear coordinate transformation. Rev. Sci. Instr. **26**, 23–27 (1955).

***Strachey, C. S. Logical or non-mathematical programmes.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 46–49. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Wilkes, M. V. Pure and applied programming.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 121–124. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Demuth, Howard B., Jackson, John B., Klein, Edmund, Metropolis, N., Orvedahl, Walter, and Richardson, James H. MANIAC.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 13–17. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Codd, E. F., and Herrick, H. L. Input scaling and output scaling for a binary calculator.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 21–23. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Perry, C. L. The logical design of the Oak Ridge digital computer.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 23–27. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Dobbins, Willis E. Designing a low cost general purpose computer.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 28–29. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Moore, E. F. A simplified universal Turing machine.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 50–54. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Reed, Irving S. Symbolic synthesis of digital computers.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 90–94. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Estrin, Gerald. A description of the electronic computer at the Institute for Advanced Studies.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 95–109. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Chu, J. C. The Oak Ridge automatic computer.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 142–148. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

***Johnston, R. F. The University of Toronto model electronic computer.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 154–160. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D. C., 1953.

Džems-Levi, G. E. Projective transformations and nomograms. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 208–211 (1954). (Russian)

A circle and a tangent are used as a nomogram for the equation $x+y=z$. A scale on the circle is used for both x and y, while the scale for z is on the tangent. It is shown that the scales can be deformed by a projective transformation.
M. Goldberg (Washington, D. C.).

Kreĭnes, M. A., and Aĭzenštat, N. D. On the possibility of nomographing with accuracy up to infinitesimals of higher order. Dokl. Akad. Nauk SSSR (N.S.) **95**, 1137–1140 (1954). (Russian)

Aczél, János, Béda, Gyula, Gáti József, et Török, Sándor. Nomogrammes pour la résolution des triangles généraux. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 383–394 (6 plates) (1954). (Hungarian. Russian and French summaries)

The authors describe various nomograms and slide rules for the solution of a plane triangle, given two sides and one angle, two angles and one side, or three sides. They claim that their nomograms are easier to draw, and can be used over a wider range, than nomograms constructed by previous investigators for the same purpose. *A. Erdélyi.*

Pál, Sándor, und Tóth, Károly. Beispiele für die nomographische Darstellung symmetrischer funktioneller Zusammenhänge mittels binären Leitertafeln. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 367–381 (1954). (Hungarian. Russian and German summaries)

In the first part of this paper the authors construct a nomogram of the fifth class for

$$(1)\qquad \begin{vmatrix} f(z_1, z_0) & g(z_1, z_0) & h(z_1, z_0) \\ f(z_2, z_0) & g(z_2, z_0) & h(z_2, z_0) \\ F(z_3, z_4) & G(z_3, z_4) & H(z_3, z_4) \end{vmatrix} = 0.$$

and some related nomograms. Taking account of the symmetry of (1), a double-grid nomogram may be constructed for this equation, one of the grids representing (z_3, z_0), and the other representing both (z_0, z_1) and (z_0, z_2). The second part of the paper contains some practical applications to problems in electrical engineering. *A. Erdélyi.*

Nygaard, K. Calculation by nomograms of the astronomical correction to precise levelling. Geodaetisk Inst. Skr. (3) **20**, 1–13 (11 plates) (1954).

RELATIVITY

***Einstein, A., et Kaufman, B. Sur l'état actuel de la théorie générale de la gravitation.** Louis de Broglie, physicien et penseur, pp. 321–336. Editions Albin Michel, Paris, 1953. 870 francs.

This paper contains a short and clear description of the Einstein-Schrödinger method (which in the meantime became common knowledge among experts) leading to the four sets of Einstein's conditions for the unified field theory. In the second part of the paper the "strong" case

$$R_{(ij)}=0 \tag{1}$$

is considered. Using the method of successive approximations, the authors find the restrictions imposed on the field g_{ij} in the case of the Schwarzschild solution $g_{(ij)}$. This result "semble exclure d'un point de vue physique toute utilité du 'système fort'." *V. Hlavatý* (Bloomington, Ind.).

***Einstein, A. Appendice.** Louis de Broglie, physicien et penseur, pp. 337–342. Editions Albin Michel, Paris, 1953. 870 francs.

If one applies the Schrödinger transformation

$${}^{0}\Gamma^{i}_{jk}=\Gamma^{i}_{jk}+\tfrac{2}{3}\delta_j{}^{i}\Gamma^{s}_{[ks]} \tag{1}$$

after dealing with the usual variational problem of the appropriate curvature density, one obtains all four sets of Einstein conditions for the unified field g_{ij}. The transformation (1) may be generalized to

$${}^{*}\Gamma^{i}_{jk}=\Gamma^{i}_{jk}+\delta_j{}^{i}\lambda_k \tag{2}$$

where λ_k is an arbitrary vector. Some elementary geometrical properties of (2) are dealt with, as for instance the transformation of covariant derivatives, of curvature tensor and so on. [Remarks of the reviewer: The transformation (2) preserves obviously the parallelism. The connection (1) is invariant with respect to (2)]. *V. Hlavatý.*

Winogradzki, Judith. Sur les λ-transformations de la théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **239**, 1359–1361 (1954).

The equation (2) of the foregoing review may be generalized to

$${}^{*}\Gamma^{i}_{jk}=\Gamma^{i}_{jk}+\Lambda^{i}_{jk}, \tag{1}$$

A being a tensor. The author announces (without proof) the following result: Regardless of the kind of variation of the basic curvature density, one obtains results invariant with respect to (1) if and only if either $\Lambda^{i}_{jk}=\delta^{i}_{j}\lambda_k$ or $\Lambda^{i}_{jk}=\delta_k{}^{i}\lambda_j$. The proof will appear later. *V. Hlavatý.*

Mavridès, Stamatia. Solution non statique à symétrie sphérique des équations de la théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **239**, 1597–1599 (1954).

The author shows that under the restriction $g_{23}=0$, the only spherically symmetric solutions of the unified field equations of Einstein and Straus [Ann. of Math. (2) **47**, 731–741 (1946); MR **8**, 412] are the static solutions found by Papapetrou [Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948); MR **10**, 580]. *F. A. E. Pirani* (Dublin).

Papapetrou, Achilles. Eine neue Theorie des Gravitationsfeldes. I, II. Math. Nachr. **12**, 129–141, 143–154 (1954).

I. The author assumes

(1) a) $ds^2=-e^{\varphi}(dx^2+dy^2+dz^2)+e^{\chi}dt$

b) $\mathfrak{T}^{\nu}_{\mu;\nu}=0$ ($\mathfrak{T}$ = momentum-energy tensor)

and deals with the variational problem of

$$\mathfrak{g}^{\alpha\beta}\left[\begin{Bmatrix}\mu\\ \alpha\nu\end{Bmatrix}\begin{Bmatrix}\nu\\ \beta\mu\end{Bmatrix}-\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}\begin{Bmatrix}\nu\\ \mu\nu\end{Bmatrix}\right]+\kappa g_{\mu\nu}\mathfrak{T}^{\mu\nu} \tag{2}$$

by means of (1)a. He obtains two field equations

$$R=-\kappa T_{\alpha}{}^{\alpha},\quad 2R_0{}^{0}=\kappa(T_0{}^{0}-T_i{}^{i}) \tag{3}$$

$$(\alpha,\beta,\cdots=0,\cdots,3;\ i,j,\cdots=1,2,3)$$

for two unknowns φ, χ.

If one requires that the static spherically symmetric solution could be prolonged into the central body, one obtains outside of it the Schwarzschild solution

$$ds^2=-\left(1+\frac{\alpha}{r}\right)^4(dx^2+dy^2+dz^2)+\left(\frac{1-\alpha/r}{1+\alpha/r}\right)^2dt, \tag{4}$$

so that by virtue of the assumption (1)b this theory yields the Einstein results concerning the three famous phenomena of celestial mechanics. For a weak field, $|\varphi|$, $|\chi|\ll 1$, the equations (3) reduce to a Poisson-like system which does not admit in vacuo any plane wave solution.

II. The requirement of covariance of the second equation (3) is satisfied by trivial transformations $x'\to x$ with $\partial x'^{0}/\partial x^{i}=\partial x'^{i}/\partial x^{0}=0$ and

$$e^{\varphi}(dx^2+dy^2+dz^2)=e^{\varphi'}(dx'^2+dy'^2+dz'^2)$$

which are uninteresting from the point of view of physics. The more general transformations with $\partial x'^{0}/\partial x^{i}\neq 0$ require two conditions to be imposed on $\varphi-\chi$ which are satisfied in the case (4). But even then the corresponding infinitesimal transformations $x'\to x+\epsilon\xi$ with $\xi^{i}_{,0}\neq 0$ do not yield any generalization of Lorentz transformations. The author investigates in detail the infinitesimal transformations together with the boundary conditions φ, $\chi\to 0$ for $r\to\infty$ and shows the incompatibility of this latter with the assumption $\xi^{0}_{,i}\neq 0$ (for $\varphi\neq\chi$). Hence a not overdetermined gravitational field defines an absolute coordinate system in which (1)a holds. *V. Hlavatý* (Bloomington, Ind.).

Hennequin, Françoise. Remarques en théorie unitaire de Thiry. C. R. Acad. Sci. Paris **239**, 1582–1583 (1954).

The author defines a system of isothermal coordinates in a space V_5 as those in which

$$\Phi^{\lambda}=-\gamma^{\alpha\beta}\Gamma^{\lambda}_{\alpha\beta}=\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^{\alpha}}(\gamma^{\alpha\lambda}\sqrt{\gamma})=0,$$

where $\gamma^{\alpha\beta}$ are the contravariant components of the metric tensor, $\Gamma^{\lambda}_{\alpha\beta}$, the components of the Christoffel symbols, and γ is the determinant of the metric tensor. He evaluates Φ^{λ} for the space of projective relativity and observes that $\Phi^{i}=0$ is a generalization of the requirement that a coordinate system be isothermal in the four-dimensional space-time and that the condition $\Phi^{0}=0$ is a generalization of the Lorentz condition which partially determines the gauge frame in electro-magnetic theory. The second part of the paper is concerned with an interpretation of the conservation theorems which follow from the field equations proposed by Y. Thiry in projective relativity. *A. H. Taub.*

Just, Kurt. Zur Wahl von Feldgleichungen der projektiven Relativitätstheorie. Z. Physik **139**, 498–503 (1954).

The author proposes certain specific values for the two parameters entering in the Lagrangean function from which the field equations of Jordan's formulation of projective relativity with a variable "gravitational constant" are derived. *A. H. Taub* (Urbana, Ill.).

Scherrer, Willy. Zur linearen Feldtheorie. I. Ein Wirkungsprinzip und seine Anwendung in der Kosmologie. Z. Physik **139**, 44–55 (1954).

The author gives a determination of the arbitrary function $L(x_0)$ entering in the line element of a cosmological space

$$ds^2 = dx_0^2 - L^2(x_0)(d\Theta^2 + \sin^2\Theta\, d\theta^2 + \sin^2\Theta \sin^2\theta\, d\phi^2)$$

by a generalization of a variational principle given in a previous paper [Z. Physik **138**, 16–34 (1954); MR **16**, 79]. The connection between the theory developed in this paper and the standard theory of relativistic cosmology is not discussed. In particular, the relation between the stress-energy tensor defined by the author and the metric tensor of the space-time is different from that occurring in the latter theory, and this difference is ignored. Until the relation between these stress energy-tensors is clarified the physical interpretation given to certain constants of integration occurring in this paper would seem to be questionable. *A. H. Taub* (Urbana, Ill.).

Takeno, Hyôitirô. On solutions of electromagnetic equation in non-static spherically symmetric space-times. Tensor (N.S.) **4**, 9–15 (1954).

Wave solutions of Maxwell's equations in relativity are given for the general spherically symmetric form of space-time. The results generalise those given by Y. Ueno for static spherically symmetric space-time. *A. G. Walker.*

Graef Fernández, Carlos. Movement of a mass which annihilates its own gravitational field. Rev. Mexicana Fis. **3**, 253–267 (1954). (Spanish)

This paper is based upon Birkhoff's theory of gravitation [Bol. Soc. Mat. Mexicana **1**, nos. 4 and 5, 1–23 (1944); MR **6**, 240; for a survey of the theory, with bibliography, see Graef Fernández, Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, 1951, Montevideo, Uruguay, 1952, pp. 121–137; MR **14**, 807]. The author's present work is based upon a formula for the gravitational tensor h_{ij} given by him in Rev. Mexicana Fis. **1**, 11–27 (1952) [MR **14**, 418]. His problem is to investigate whether a test-particle m can remain at rest at a point O in the gravitational field of a massive particle M which moves in a straight line through O. He finds that m will remain at rest if M starts from O with the velocity of light and with a suitably retarded motion, in due course coming to rest and moving back towards O with an accelerated motion. On the return journey it attains the velocity of light before reaching O, the outward return motions being asymmetric. *H. S. Ruse* (Leeds).

Gomes, Ruy Luís. On the notion of distance in restricted relativity. Gaz. Mat., Lisboa **15**, no. 57, 3–4 (1954). (Portuguese)

L'A. considère une "distance relative" définie de la manière suivante: un repère admissible étant choisi, soit M un point matériel en mouvement rectiligne uniforme de vitesse u et soit P et Q deux points matériels en mouvement de vitesses égales v. La distance $d_u{}^v$ des points P et Q relativement à l'espace propre de M est donnée par

$$(d_u{}^v)^2 = \sum \Delta' x_i^2 - c^2 \Delta' t^2,$$

où $\Delta' x_i$, $\Delta' t$ correspondent aux positions de P et Q situés sur une direction orthogonale à la ligne d'univers de M. Différents cas particuliers et applications sont étudiées. *A. Lichnerowicz* (Paris).

MECHANICS

Froda, Alexandru. Sur les fondements de la mécanique des mouvements réalisables du point matériel. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **3**, 321–365 (1952). (Romanian. Russian and French summaries)

This is the main paper in a sequence devoted to the axioms of particle mechanics. Two of the others have already been reviewed [same Acad. Bul. Ști. Secț. Ști. Mat. Fiz. **3**, 157–175, 435–440 (1951); **15**, 692, 835]. The author considers the usual approach inadequate because it assumes the existence of velocity and acceleration. He proposes axioms which "lend themselves to experimental verification." He takes the displacement as continuous and subject to seven postulates of a mechanical nature and three postulates of finitude (pp. 328–331). The author shows familiarity with much of the classical literature on the foundations of mechanics, and this work appears to be a serious and competent effort to make use of the theory of real functions in mechanics. The reviewer's knowledge of Romanian is not sufficient to understand much of what the author writes, and he regrets that the author has not chosen a language accessible to a greater proportion of specialists in mechanics. Essential to the author's treatment is the notion of "realisable" motion, but the reviewer cannot understand its definition as "a real motion or the limit of a sequence of real motions in which the passive resistances approach zero," (pp. 322–323), since "passive resistance" is not defined. The author's postulates apply only to realisable motions. He proves a number of theorems regarding motions satisfying his axioms: for example, velocity exists and is of bounded variation, mean acceleration is bounded, the upper and lower derivatives of velocity exist. He characterizes the possible discontinuity in the acceleration. This study is of interest in that the mechanics of impacts is automatically included without any special distinction. Nevertheless its emphasis on the type of analysis found in works on functions of a real variable may make it seem unnatural and ugly to some of those who would otherwise be interested in its results. *C. Truesdell* (Bloomington, Ind.).

Moroškin, Yu. F. Foundations of the analytic theory of mechanisms. Trudy Sem. Teorii Mašin i Mehanizmov **14**, no. 54, 25–50 (1954). (Russian)

Zinov'ev, V. A. Design of spatial four-link mechanisms according to a complete set of parameters. Trudy Inst. Mašinoved. **14**, no. 55, 49–62 (1954). (Russian)

This is an independent solution to a set of problems similar to those considered by Levitskiĭ and Šahbazyan [same Trudy **14**, no. 54, 5–24 (1954); MR **16**, 532]. *M. Goldberg* (Washington, D. C.).

Hiersig, H. M. Geometrie und Kinematik der Evolventenschnecke. Forsch. Gebiete Ingenieurwesens **20**, 178–190 (1954).

The surface of the involute worm is generated by the helical motion of a straight line. The equations of the surface and the mating surface of the worm wheel, the transverse and tangential sections are derived. The points of engagement lie on the normals which pass through the axis, and their totality is the surface of engagement. Its intersections with the worm surface are the lines of engagement. The number of simultaneous lines of engagement depends upon the directions of the normals and the outlines of the worm and wheel. The curvatures of the various parts are determined by the use of the Euler-Savary formula and the Bobillier construction. *M. Goldberg.*

Peres, N. J. C. Geometry of involute helicoidal hob and gear teeth. Austral. J. Appl. Sci. **5**, 309–329 (1954).

The equations of the involute worm and its sections are derived in a manner similar to that of the paper reviewed above. *M. Goldberg* (Washington, D. C.).

***Schmid, Wilhelm. Konstruktion der Doppelpunkte einer Koppelkurve.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 271–279. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

The coupler curves traced by points on the connecting-rod joining two cranks are sextic curves with three double-points. The triangle F made by the three foci (two of the foci are the axes of the cranks) is similar to the triangle of the tracing point and the ends of the connecting-rod. The triangle F and the triangle D of the double-points lie in an isogonal position on the same focal circle. The isogonal inverse of any line with respect to D is a conic section which circumscribes D. By constructing this conic and its intersections with the focal circle, the double-points are found. Eckhart [Maschinenbau **15**, 697–698 (1936)] used a hyperbola for this conic. The author uses a parabola also. In addition, the construction considers isolated and imaginary double-points. *M. Goldberg* (Washington, D. C.).

Rosenauer, N. Complex variable method for synthesis of four-bar linkages. Austral. J. Appl. Sci. **5**, 305–308 (1954).

In the equation $ae^{i\theta_1}+be^{i\theta_2}-ce^{i\theta_3}-de^{i\nu}=0$, which expresses the fact that a plane four-bar linkage is closed, each term represents the length and position of a bar. By differentiating twice with respect to time, two other equations involving velocities and accelerations are derived. If we are given the desired instantaneous velocities and accelerations, the sought linkage is obtained by solving these three equations for the lengths and angles of the bars. Reference is made to S. Bloch [Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1940**, no. 1, 47–54 (1940)] for this method.

M. Goldberg (Washington, D. C.).

Rózsa, Pál. Untersuchungen über kleine Schwingungen elastisch gekoppelter Korpuskularsysteme unter Heranziehung des Matrizenkalküls. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 51–82 (1 plate) (1954). (Hungarian. Russian and German summaries)

The author wishes to show in this paper that matrices can be used with advantage in the theory of small oscillations of systems of particles. He discusses the following examples: (i) a system of $2n$ particles in which particles of masses m, M alternate, each particle being coupled to its first neighbours (model of a vibrating string with periodically varying mass distribution); (ii) transversal vibrations of a system of n equal particles placed at equal distances along a circular cross-section of a smooth cylinder (this illustrates the case of double latent roots); (iii) the increase in energy of a system of particles under the influence of periodically repeated impulses of equal strength.

A. Erdélyi (Pasadena, Calif.).

Caprioli, Luigi. Sul comportamento energetico di alcuni sistemi meccanici non-lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 463–467 (1954).

The equation $m\ddot{x}+h^2\dot{x}+k^2x+F(v_0+\dot{x})=0$ can be interpreted as the equation of motion of a particle subject to viscous friction, a restoring force applied by a spring, and friction applied by a belt which moves with the speed v_0. In this note the author considers periodic oscillations of the particle, and gives some elementary calculations of the work done by the motor driving the belt, and of the work done on the particle by the belt. *L. A. MacColl.*

Greenspan, Martin. Simple derivation of the Boltzmann-Ehrenfest adiabatic principle. J. Acoust. Soc. Amer. **27**, 34–35 (1955).

If an harmonic oscillator is vibrating with natural frequency ω and energy E and if a small slow change in some of the "constants" of the system is made, then it follows from the Boltzmann-Ehrenfest Principle that the change in energy δE and the change in frequency $\delta\omega$ are related by (*) $\omega\delta E=E\delta\omega$. The author derives (*) for this situation in a simple manner from Lagrange's equations. The time-dependant generalized force causing the change is also given. Finally, the differential equation of an inductance-capacitance circuit with slightly varying inductance and capacitance is considered; the approximate solution shows that (*) is again satisfied. *H. D. Block* (Minneapolis, Minn.).

Lawden, D. F. Stationary rocket trajectories. Quart. J. Mech. Appl. Math. **7**, 488–504 (1954).

In a previous paper [J. Amer. Rocket Soc. **23**, 360–367 (1953)] the author calculated the fundamental conditions which must be satisfied by a rocket trajectory in vacuo, of least fuel expenditure joining two given points in space. This paper extends the theory to cases involving not only a general gravitational field but also a dissipative aerodynamic force given as a function of the rocket's velocity and position in space. The simplifying assumption is made that a particular orientation of the rocket with respect to its direction of motion is automatically maintained during flight. The external forces are thus assumed to be known functions of t, x_i, dx_i/dt, and M (the instantaneous mass) only ($i=1, 2, 3$). By changes of variables, dropping of higher order terms, and explicit integration, conditions are obtained first for stationary trajectories, where thrust is maintained throughout the trajectory and an impulsive force may occur at one point. Null-thrust arcs are handled by passage to a limit. The chief conclusions are in terms of boundary values and continuity of certain complicated auxiliary functions. An application is made to the case of optimum programming for fuel expenditure for a rocket ascending vertically, to achieve a given height, assuming uniform gravity, and air resistance according to the Square Law. An explicit formula is obtained. No quantitative values are mentioned. *A. A. Bennett* (Providence, R. I.).

***Popoff, Kyrill. Die Hauptprobleme der äusseren Ballistik im Lichte der modernen Mathematik.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1954. xv+278 pp. DM 22.00.

This volume, although nothing on the chief title page so suggests, is a second edition of an earlier work [Leipzig, 1932]. The present volume consists of three parts. I. The motion of a point in an atmosphere with constant density (pages 11–137). II. The motion of a point in an atmosphere of variable density (two density laws are considered) (pages 138–211). III. The motion of the projectile about its centroid (pages 212–274). The first two parts, reproduced here without any alteration, constituted the first edition, while the third part is new. Throughout the entire work, the trajectory is assumed to be plane and the differential equations exact, with an analytic (but unidentified) drag function. The author is specially interested in treating the ballistic equations by general mathematical methods and therefore studies power-series expansions, series solutions of integral equations, singularities for $v=\infty$, $y'=\infty$, $x=\infty$, etc. Part III examines precession and nutation, but no effects of lift or kiting. Stability is only touched upon. The need for further aerodynamic research, and the difficulties incident thereto for the actual velocities experienced, are emphasized in fitting manner. The fact that the temperature of the air is not constant with altitude is ignored, nor are shock waves mentioned. The author's interests lie not in the direction of practicable procedures for constructing a firing table, but in manipulating various forms of an idealized set of ballistic equations to arrive at formally convergent solutions.

A. A. Bennett (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

***Levič, V. G. Fiziko-himičeskaya gidrodinamika. [Physical-chemical hydrodynamics.]** Izdat. Akad. Nauk SSSR, Moscow, 1952. 538 pp. 23.60 rubles.

This book treats aspects of hydrodynamics with which, for the most part, a mathematician is not usually concerned, perhaps because they seldom come to his attention. After an introductory chapter outlining the standard hydrodynamic theory, especially viscous flow, boundary-layer theory, and turbulent flow, the following topics are treated: convective diffusion; diffusion in a turbulent flow; passage of current through a solution of electrolytes; capillary motion; motion of drops and bubbles; motion of particles in solutions of electrolytes; theory of the polarograph; waves on the surface of a fluid; motion and diffusion in thin layers of a liquid. Each chapter is followed by a bibliography (surprisingly, the work of Gromeka on the motion of drops [Sobranie sočinenil, pp. 206–230, Akad. Nauk SSSR, Moscow, 1952; MR **15**, 89] is not listed). The mathematical methods are elementary and interest for a mathematician will lie chiefly in the subjects treated. *J. V. Wehausen.*

Pratelli, Aldo M. Principi variazionali nella meccanica dei fluidi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 484–500 (1953).

This is a leisurely discussion of general properties of the kinetic energy of a continuous medium. The author remarks that Kelvin's theorem of minimum energy can be extended to steady motions of a compressible substance, providing the density is not varied. For an incompressible substance, among all steady motions having assigned vorticity (possibly zero) and also the same value of the tangential component of velocity on the boundary, that which satisfies the continuity equation yields the smallest kinetic energy. The paper concludes with another similar theorem which does not seem to the reviewer to be meaningful as stated since it refers to the vector and scalar potentials as if they were uniquely determined. *C. Truesdell.*

Pratelli, Aldo M. Sulla stazionarietà di significativi integrali nella meccanica dei continui. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 714–724 (1953).

The author remarks that if a symmetric tensor is split into a "potential" part $\phi_{(i,j)}$ plus a solenoidal part, then for it itself to be solenoidal it is necessary and sufficient that the spatial mean of its square be a minimum with respect to variation of the potential part only. A similar result holds in four dimensions. By verbal alteration, these theorems can refer to the equilibrium and motion of continuous media. The author obtains a similar variational expression for the conditions of compatibility for infinitesimal strains. He shows the connections between his results, the theorem of minimum energy, and the theorem of Menabrea.

C. Truesdell (Bloomington, Ind.).

Backes, F. Sur les lames liquides en équilibre. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 30–34 (1955).

L'auteur expose certains résultats classiques concernant les lames liquides en équilibre et affectant la forme d'une surface à courbure moyenne constante et montre le lien qui existe entre ces résultats et une formule de géométrie différentielle. *F. Şemin* (Istanbul).

Singh, K. R. Path of a vortex round the rectangular bend of a channel with a uniform flow. Z. Angew. Math. Mech. **34**, 432–435 (1954).

This paper considers the path of a single rectilinear vortex in a uniform flow through a two-dimensional L-shaped channel (precisely, the channel consists of those points (x, y) which satisfy $0\leqq x\leqq h$, $0\leqq y\leqq\infty$, or $h\leqq x\leqq\infty$, $0\leqq y\leqq h$, and the basic flow is uniform and of speed U at both "ends"). The paths of the vortex are found by means of the standard techniques of conformal mapping. They are given parametrically in a closed form, and several of the possible situations are illustrated. It is interesting that the behavior of the vortex is different in the three cases $\alpha\leqq -2$, $-2\leqq\alpha<0$, $0<\alpha$, where $\alpha=\Gamma/4Uh$ and Γ=circulation.

J. B. Serrin (Minneapolis, Minn.).

Cummins, William E. Hydrodynamic forces and moments acting on a slender body of revolution moving under a regular train of waves. The David W. Taylor Model Basin, Washington, D. C., Rep. **910**, vi+33 pp. (1954).

The hydrodynamical force and moment acting on a slender body of revolution are found for the case in which the body is moving with constant linear velocity under a sinusoidal train of waves oblique to the course of the body. The analysis makes use of a representation of the body by a system of singularities and the dynamic effects are evaluated by means of the methods developed by the author in Taylor Model Basin Rep. **780** (1953) [MR **15**, 70]. The force and moment are given explicitly as functions of the sectional-area curve of the body. Three illustrative examples are worked out, including a case in which there is no analytical expression available for the sectional-area curve. The results constitute a linearized approximation. They are of great practical interest. *L. M. Milne-Thomson* (Greenwich).

Keller, Joseph B., and Kolodner, Ignace. Instability of liquid surfaces and the formation of drops. J. Appl. Phys. **25**, 918–921 (1954).

The theory of Taylor instability [G. I. Taylor, Proc. Roy. Soc. London. Ser. A. **201**, 192–196 (1950); MR **12**, 58; R. Bellman and R. H. Pennington, Quart. Appl. Math. **12**, 151–162 (1954); MR **16**, 83] is extended to the case of a liquid layer of thickness h between two parallel free boundaries (e.g., of gas) at different pressures p_1, p_2. The most unstable perturbation mode is then calculated, with surface tension T included, and used to predict a mean drop radius r under breakup. The final formula is $r=[9\pi Th^2/2|p_2-p_1|]^{1/3}$; no comparisons with experimental data are given. *G. Birkhoff* (Cambridge, Mass.).

Subba Rao, Ramachandra, and Nigam, Swami Dayal. The effect of cross-viscosity on the performance of full journal bearing without side-leakage. Z. Angew. Math. Phys. **5**, 426–429 (1954).

The authors suggest that the discrepancy between experiment and Sommerfeld's theory of lubrication is due not to side leakage but to cross viscosity [cf., e.g., Ch. V C of the reviewer's paper, J. Rational Mech. Anal. **1**, 125–300 (1952); **2**, 593–616 (1953); MR **13**, 794; **15**, 178]. They set up the equations for plane flow of an incompressible fluid with quadratic viscosity. First they neglect the inertia of the fluid, then they get an "approximate solution" by neglecting some other terms. Their final result is adjustable by means of a parameter of cross-viscosity. [The reviewer is impelled to call attention to the capricious selection of some nonlinear terms while others are neglected.] *C. Truesdell.*

Szebehely, V. G., and Pletta, D. H. The analogy between elastic solids and viscous fluids. Bull. Virginia Polytech. Inst. (Eng. Exp. Sta. Ser. No. 80) **45**, ii+24 pp. (1951).

The main purpose of this paper seems to be to point out a number of formal similarities between the equations of linear elasticity and the equations for compressible viscous fluids for which the stress is assumed to be a linear function of the rate of deformation. Anisotropic and isotropic, homogeneous and inhomogeneous materials are considered. Brief expositions of the kinematics of finite strain and rate of deformation are included. Though the authors purport to use curvilinear coordinates, several of their relations are not tensorially invariant under general coordinate transformations. For example, their "displacement vector" is neither a covariant nor contravariant vector under general transformations. This type of error is eliminated if one agrees always to refer undeformed and deformed positions to the same rectangular Cartesian coordinate system.

J. L. Ericksen (Washington, D. C.).

Wuest, W. Strömung durch Schlitz- und Lochblenden bei kleinen Reynolds-Zahlen. Ing.-Arch. **22**, 357–367 (1954).

By Stokes' approximation, viscous flows through a slit and a hole are calculated by introducing elliptic, spherical and ellipsoidal coordinate systems. In the plane case, if the flow is radial, it is calculated for moderate Reynolds numbers as well by extending the range of the variable in the well-known divergent channel flows. *Y. H. Kuo.*

Bhagavandin, Kettarnath. On the motion of circular cylinders through viscous incompressible fluids. Norske Vid. Selsk. Forh., Trondheim **27**, no. 14, 6 pp. (1954).

In this paper the uniform motion of a circular cylinder in a viscous, incompressible fluid enclosed by a circular cylinder of larger radius is considered. Both the first- and second-order approximations to the stream function of the motion are obtained, and are expressed in compact form.

R. C. Di Prima (Cambridge, Mass.).

Skavlem, S., and Tjötta, S. Steady rotational flow of an incompressible, viscous fluid enclosed between two coaxial cylinders. J. Acoust. Soc. Amer. **27**, 26–33 (1955).

By successive approximations at low Reynolds number, the author calculates the unsteady viscous flow between coaxial cylinders of which the inner one oscillates harmonically in a plane. To the second approximation, the results show that there exist two stationary circulatory systems in every quadrant and that the direction of circulation is opposite in these systems. The numerical results compare favorably with experiments. *Y. H. Kuo.*

Lambossy, Paul. Oscillations forcées d'un liquide incompressible et visqueux dans un tube rigide et horizontal. Calcul de la force de frottement. Helv. Phys. Acta **25**, 371–386 (1952).

This paper considers the unsteady viscous flow through a straight circular pipe by maintaining a harmonically oscillating pressure difference. It is shown that the shear stress at the wall varies harmonically with a phase shift and can be expressed linearly in terms of the mean velocity and acceleration with coefficients depending on the radius of the pipe, the fluid properties and the reduced frequencies.

Y. H. Kuo (Ithaca, N. Y.).

Sexl, Theodor. Über eine Eigenschaft der Poiseuilleschen Strömung. Acta Phys. Austriaca **9**, 75–76 (1954).

A flow in a straight pipe will be called "steady laminar" if the velocity components are of the form $u=v=0$, $w=w(x, y)$, where (x, y, z) are rectangular coordinates with the z-axis parallel to the pipe. The author proves the following theorem: The steady laminar flow of an incompressible viscous fluid down a straight pipe of circular cross-section (Poiseuille flow) has the property that its energy dissipation is least among all steady, laminar and axially symmetric flows down the pipe which have the same total flux and which do not slip at the walls.

The reviewer notes that the condition of axial symmetry may be dropped, and that the same theorem is true regardless of the shape of the cross-section. These results are in essence contained in the theorems of Helmholtz and Rayleigh on "slow" motion of viscous fluids [cf. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, pp. 617–618].

J. B. Serrin (Minneapolis, Minn.).

Stewartson, K. On the motion of a flat plate at high speed in a viscous compressible fluid. I. Impulsive motion. Proc. Cambridge Philos. Soc. **51**, 202–219 (1955).

The author investigates the viscous compressible flow produced by moving impulsively an infinite plate parallel to itself with constant speed. When the Mach number is high, it is assumed that there exists a shock advancing laterally, and that following the shock the flow can be subdivided into inviscid and viscous regions. It is then established that, for some small time t, similar solutions are possible for both regions. The results confirm those obtained by boundary-layer theory for semi-infinite plate in steady flow. *Y. H. Kuo* (Ithaca, N. Y.).

Loos, Henk G. A simple laminar boundary layer with secondary flow. J. Aero. Sci. **22**, 35–40 (1955).

The incompressible laminar boundary layer over a flat plate is studied for the case in which the stream lines in the free flow have a parabolic shape. The velocity distribution in the free stream is such that there is no pressure gradient normal to the leading edge. In this case, an exact solution of the boundary-layer equations can be obtained; the boundary-layer velocity components are expressed in terms of the well-known Blasius solution and a function tabulated by Mager and Hansen [NACA Tech. Note no. **2658** (1952)]. Velocity components normal and tangential to the free flow direction are computed. (The boundary-layer velocity component normal to the free stream is known as the cross flow or secondary flow.) It is found that in certain cases even though "separation" type profiles for the tangential flow are obtained, there will be no separation because of the action of the secondary flow. Finally it is shown that the total pressure at a given point in the boundary layer can exceed that of the corresponding free stream. *R. C. Di Prima.*

Görtler, H. Decay of swirl in an axially symmetrical jet, far from the orifice. Rev. Mat. Hisp.-Amer. (4) **14**, 143–178 (1954).

In this paper the problem of the decay of a swirl in an axially symmetric jet (swirl refers to the velocity component of rotation about the axis of the jet) is treated. Assuming that the swirl is small, the equations of motion for a laminar jet are linearized and an eigenvalue problem of an ordinary differential equation of the second order is obtained. By a suitable change of variables the differential equation is reduced to that for the Legendre functions. It is shown that the swirl in a free jet always decreases according to a higher power of the inverse distance from the orifice than the velocity component of the jet in the direction of the axis. The problem of a turbulent free jet with a weak swirl is also discussed. *R. C. Di Prima* (Cambridge, Mass.).

Chandrasekhar, S. The character of the equilibrium of an incompressible heavy viscous fluid of variable density. Proc. Cambridge Philos. Soc. **51**, 162–178 (1955).

The problem considered in this paper is the following: Given a static state in which an incompressible, viscous fluid is arranged in horizontal strata and the density is a function only of the vertical coordinate z, to determine the initial manner of development of an infinitesimal disturbance. A system of equations for the three velocity components u, v, and w (assumed small) and the perturbation density and pressure are derived. It is assumed that the functions are of the form $w=w(z)\exp(ik_x x+ik_y y+nt)$, and a fourth-order eigenvalue problem is obtained. Appropriate boundary conditions for a rigid or a free surface are stated.

The particular case of two uniform fluids of different densities (but the same kinematic viscosity) separated by a horizontal boundary is considered. In the case that the upper fluid is more dense, curves of n vs. k ($k^2=k_x^2+k_y^2$) are given for various value of $(\rho_2-\rho_1)/(\rho_1+\rho_2)$, and the modes of maximum instability are stated. Here 2 refers to the upper fluid, and 1 to the lower. For $\rho_2<\rho_1$ it is found that for $k<k_*$ (k_* depends on $(\rho_1-\rho_2)/(\rho_1+\rho_2)$) the decay, is an oscillation of decreasing amplitude, and for $k>k_*$ there are two modes of decay which are both aperiodic. The limiting case of gravity waves (when $\rho_2=0$) is also considered.

Finally a variational principle (which may be used in approximate treatments of more general problems) is derived. Several observations concerning the character of the eigenvalue, n, are made. *R. C. Di Prima.*

Hide, Raymond. The character of the equilibrium of an incompressible heavy viscous fluid of variable density: an approximate theory. Proc. Cambridge Philos. Soc. **51**, 179–201 (1955).

In this paper a variational principle derived by S. Chandrasekhar (see the above review) is used to obtain approximate solutions of the following equilibrium problems. The first is that of two superposed fluids of great depth, and densities and viscosities ρ_2, ρ_1 and μ_2 and μ_1 (the notation here is the same as in the above review). In the particular case, $\mu_1/\rho_1=\mu_2/\rho_2$, the results of the approximate method are in good agreement with the exact numerical solution given by Chandrasekhar. The second is that of two superposed fluids of small depth. In this case only "long gravity waves" are considered. Finally, a continuously stratified fluid of finite depth d in which $\rho(z)=\rho_0\exp(\beta z)$ is treated under the assumption that $(\beta d)\ll 1$. In both the first and third problems stable and unstable modes are discussed in some detail. *R. C. Di Prima* (Cambridge, Mass.).

Chandrasekhar, S. The thermal instability of a fluid sphere heated within. Phil. Mag. (7) **43**, 1317–1329 (1952).

In this paper the problem of the thermal instability of an incompressible fluid sphere heated within and in equilibrium under its only gravitation is considered. The basic equations of the theory are derived, as well as the appropriate boundary conditions that must be satisfied on rigid and free spherical surfaces. In the case of marginal stability the disturbance is analysed into modes in terms of spherical harmonics of various orders l; and a variational principle is derived for the characteristic number c_l which governs the onset of convection. c_l is computed for the first fifteen modes both when the bounding surface is free and when it is rigid; and it is shown that in both cases the mode $l=1$ is the first to be excited. (Taken in part from the author's summary.) *R. C. Di Prima* (Cambridge, Mass.).

Chandrasekhar, S. The onset of convection by thermal instability in spherical shells. Phil. Mag. (7) **44**, 233–241; correction, 1129–1130 (1953).

In this paper the variational principle derived by the author in the paper reviewed above is used to consider the problem of the thermal instability of an incompressible sphere consisting of an inviscid core and a viscous mantle. It is shown that the pattern of convection which sets in, at marginal stability, in the mantle shifts to harmonics of the higher orders as the thickness of the mantle decreases. Thus, when the mantle extends to a depth of half the radius of the sphere, the harmonics of order three and four set in about simultaneously, while the harmonic of order five follows very soon afterwards. The bearing of this result on the problem of convection in the earth's mantle and of the interpretation of the earth's topographic features is indicated. The numerical computations are slightly incorrect in the first paper but are corrected in a succeeding note (see the title). The general conclusions are not affected by this correction. (Taken in part from the author's summary.)

R. C. Di Prima (Cambridge, Mass.).

Lessen, M. Note on the propagation of infinitesimal disturbances in gases according to the Navier-Stokes equations. J. Aero. Sci. **21**, 849–850 (1954).

The author derives the linearized partial differential equation satisfied by an arbitrary density disturbance in perfect gas with linear viscosity and heat conduction. This is an equation of fifth order. He remarks that shear modes propagate independently of longitudinal disturbances. [The author's attributions are misleading. He does not "generalize to any stress disturbance" the analysis of the reviewer [J. Rational Mech. Anal. **2**, 643–741 (1953); MR **15**, 757], which concerned the solution of the characteristic equation. The author's remarks on the isothermal and isentropic limits are essentially the same as those of Lamb, which are repeated on p. 659 of the reviewer's paper.]

C. Truesdell (Bloomington, Ind.).

Phillips, O. M. The irrotational motion outside a free turbulent boundary. Proc. Cambridge Philos. Soc. **51**, 220–229 (1955).

L'auteur se propose de décrire un écoulement turbulent constitué de la façon suivante: le fluide occupe la région $x_1 \geqq X$ ($X<0$ donné). La vitesse u dérive d'un potentiel ϕ dans toute la région $\mathfrak{R}(x_1>0)$, et est nulle pour $x_1=\infty$. Sa composante normale au plan $x_1=0$ est une fonction aléatoire stationnaire des coordonnées x_2, x_3 dans ce plan. Par l'analyse de Fourier, on peut exprimer les composantes u_1, u_2, u_3 de la vitesse en chaque point de $\mathfrak{R}$, puis calculer les corrélations et fonctions spectrales á l'aide d'une unique fonction $\theta(\mathbf{k})$. Entre ces grandeurs existent diverses relations indépendantes de $\theta(\mathbf{k})$. En particulier $u_1^2=u_2^2+u_3^2$, et, lorsque x_1 augmente, la fonction spectrale $E(|\mathbf{k}|)$ décroît proportionnellement à $\exp\{-2|k|x_1\}$. Il est enfin possible de préciser la forme de $\theta(\mathbf{k})$ pour les petites valeurs de $\mathbf{k}$.

Une telle situation semble correspondre au mouvement en bordure de la couche limite turbulente proprement dite, région dans laquelle subsistent de petites fluctuations irrotationnelles. Les prévisions théoriques présentent un accord convenable avec les résultats expérimentaux de A. A. Townsend relatifs á la couche limite et au sillage d'un cylindre.

J. Bass (Paris).

Broer, L. J. F. On the theory of shock structure. III. Appl. Sci. Res. A. **5**, 76–80 (1954).

The profile of a weak shock wave for a gas with bulk coefficient of viscosity is compared with that of a gas with an 'equivalent' relaxation time. Equivalence is defined by the requirement that the two gases exhibit the same sound absorption at low frequencies. Using approximate theories developed in parts I and II [Appl. Sci. Res. A. **3**, 349–360 (1952); **4**, 157–170 (1954); MR **16**, 304] it is found that, if the relaxation time is large, the relaxation profile exhibits a tail not found in the purely viscous case. If the relaxation time is small, the limiting weak shock (hyperbolic tangent) profiles are identical, but, to next order, the distortion is different in the two cases, so it might be possible to observe a difference in this case also.

H. Grad.

Tricomi, Francesco G. Stranezze del "Tricomi-gas." Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 423–426 (1954).

Dans cette courte note sont établies et discutées les équations relatives au gaz fictif de Tricomi, fréquemment utilisé dans les études d'écoulements transsoniques.

P. Germain (Paris).

Kryučin, A. F. On the problem of transsonic flow about a profile. Prikl. Mat. Meh. **18**, 547–560 (1954). (Russian)

To approximate the symmetrical flow at Mach number M_∞ parallel to the x-axis with detached shock about a rhombus of small thickness τ, introduce dimensionless coordinates and velocities by $u/a^*=1-\epsilon U/(\kappa+1)$, $v/a^*=\epsilon^{1.5}V/(\kappa+1)$, $X=x/l^*$, $Y=\epsilon^{0.5}y/l^*$, where l^* denotes some reference length, a^* critical velocity of sound, κ adiabatic exponent, and $\epsilon^{1.5}=(\kappa+1)\tau$. To terms of order ϵ^3 the equations of steady inviscid plane flow with vorticity imply

$$U\partial^2 Y/\partial V^2+\partial^2 Y/\partial U^2=0. \tag{1}$$

In the UV-plane impose on Y well known boundary conditions correct to the lowest-order terms in ϵ on the approximate shock polar and on the three images of the line of symmetry, a front face of the rhombus, and the upper corner of the rhombus. Starting from a Fourier-series (in V) solution of (1) obtained by separation of variables, the author develops a method suitable for numerical computation of an approximate solution of this boundary-value problem. Approximate parametric equations are found for shock and sonic line, and the ratio of shock separation to the length of a front face of the rhombus has been computed as a function of the transonic similarity parameter $\Lambda=2(1-M_\infty)/\epsilon$.

J. H. Giese (Havre de Grace, Md.).

Ehlers, F. Edward. On some solutions of the hodograph equation which yield transonic flows through a Laval nozzle. J. Aero. Sci. **22**, 107–123 (1955).

L'auteur étudie á l'aide de l'équation de Tricomi les écoulements plans d'un fluide parfait compressible dans les tuyères; dans ce but il obtient des solutions $\psi(\theta, s)$ qui possèdent la singularité suivante: $(\partial\psi/\partial\theta)_{\psi=0}=s^{-2}$ où ψ désigne la fonction de courant, θ la direction de la vitesse et s une fonction connue de la vitesse. L'écoulement est étudié et discuté suivant les valeurs du paramètre positif μ. Ce travail recoupe plusieurs travaux antérieurs qui ne sont pas signalés dans la bibliographie [e.g., Fenain, Rech. Aéro. no. **33**, 11–28 (1953); MR **14**, 1141].

H. Cabannes.

Holt, M. A vortical singularity in conical flow. Quart. J. Mech. Appl. Math. **7**, 438–445 (1954).

L'auteur considère les écoulements coniques d'un fluide compressible dont on néglige la viscosité et la conductivité; les surfaces d'entropie constante, qui sont des cônes, possèdent une génératrice commune, appellée ligne singulière [Ferri, NACA Rep. no. **1045** (1951); MR **14**, 331]. L'auteur prouve que sur cette ligne la vitesse est radiale; il étudie l'écoulement au voisinage.

H. Cabannes (Marseille).

Aslanov, S. K. Weakly supersonic flow of an ideal gas about a thin wedge. Prikl. Mat. Meh. **18**, 561–572 (1954). (Russian)

Under the usual transonic approximations the shock in flow at Mach number M_1 with adiabatic exponent κ about a finite wedge of half angle δ becomes attached when the similarity parameter

$$K_1=(M_1^2-1)/[(\kappa+1)\delta]^{2/3}=K_*=3/2^{4/3},$$

and for $K_1=K_{1\text{im}}=2^{1/3}$ the shock becomes straight. For $K_1>K_{1\text{im}}$ the wave-drag coefficient is approximated by (1) $C_x=4(K_1-K)\delta^{5/3}/3(\kappa+1)^{1/3}$, where $K(K_1)$ is the middle root of $(K_1-K)^2(K_1+K)=2$. For $K_1<K_{1\text{im}}$ the author approximates the shock by a parabola $x=ay^2-x_a$, the sonic line by either $x=1$ or $x=1-y\delta/C$, where shock vertex curvature $2a$, shock separation x_a, and C are constants to be

determined. Now (2) $C_x = C_x^0 \delta^{5/3}(\kappa+1)^{-1/3} + C_x^*$, where C_x^0 is J. D. Cole's [J. Math. Phys. **30**, 79–92 (1951); MR **15**, 263] value at Mach 1, $C_x^* = 2\int(\Delta S/\kappa R M_1^3)\,dy$, R is the gas constant, and ΔS the change in specific entropy at the shock. The integrand of C_x^* is crudely estimated and the integration is carried only to the sonic point of the shock to obtain $C_x^* = (A_1 K_1 - A_2 K_1^2)\delta^{5/3}(\kappa+1)^{-1/3}$, where A_1 and A_2 are known functions of x_a, a, C, δ, M_1, and other undisturbed stream parameters. By requiring dC_x/dK_1 to assume the value given by Cole at Mach 1, $x_a = 0$ at $K_1 = K_*$, and equality of (1) and (2) at $K_1 = K_{1\lim}$ the author obtains three relations involving A_1 and A_2 from which x_a, a, and C can be found as functions of M_1, δ, etc. The resulting C_x agrees well with H. W. Liepman's and A. E. Bryson's [J. Aero. Sci. **17**, 745–755 (1950)] test data for $K_1 < K_{1\lim}$. To improve agreement for $K_1 > K_{1\lim}$ the author recomputes and approximates $C_x = (A_1 K_1^{-1/2} - A_2 K_1^{-2})\delta^{5/3}(\kappa+1)^{-1/3}$ by using similar approximations to $\int(\Delta S/R)\,dy$. *J. H. Giese.*

Jones, C. W. On the propagation of shock waves in regions of non-uniform density. Proc. Roy. Soc. London. Ser. A. **228**, 82–99 (1955).

L'auteur construit des solutions pour les équations des écoulements rectilignes: $x = F(t)X(\xi)$, $u = G(t)U(\xi)$, $\rho = H(t)\Omega(\xi)$, $p = K(t)P(\xi)$ où $\xi = f(t)\exp(-\alpha^{-1}kS)$, S est l'entropie, k et α des constantes. $F'G^{-1}$ est une constante d_3, si bien qu'on est ramené à une équation différentielle unique entre X et $Y = (1 - d_3 X U^{-1})^{-1}$. Un choc se propageant dans un gaz à pression constante et de densité variable $\rho_0(x)$, on peut, à l'aide des solutions précédentes, vérifier de façon exacte les équations du choc; l'énergie totale transportée par l'onde $\int_a^x \{\frac{1}{2}\rho u^2 + p/(\gamma-1)\}\,dx$ varie lentement avec le temps, si bien que l'hypothèse qui voudrait que cette énergie soit constante est approximativement vérifiée. Ces résultats complètent heureusement des résultats antérieurs dûs à G. I. Taylor [mêmes Proc. **201**, 159–174, 175–186 (1950)]. Un problème de ce type apparaît en astronomie dans l'étude du phénomène des novae. *H. Cabannes.*

Kališevič, I. Z. Approximate integration of the equation of plane vortical supersonic motion of a gas. Dokl. Akad. Nauk SSSR (N.S.) **99**, 37–40 (1954). (Russian)

Let $\psi(\sigma, \theta)$ be the stream function for a steady flow, $\sigma = p/\rho_0 V_0^2$ dimensionless pressure, θ the inclination of the velocity to the x-axis, $\lambda = V/a_*$ dimensionless speed of flow, and a_* critical speed of sound. Then

$$(1) \qquad \lambda\psi_{\sigma\sigma} - \lambda_\sigma \psi_{\theta\theta} - 2\lambda_\sigma \psi_\sigma - \lambda_\psi(\psi_\sigma)^2 - \lambda_{\sigma\psi}(\psi_\theta)^2 = 0,$$

where subscripts denote partial differentiation, and λ can be found explicitly from Bernoulli's equation as a function of σ and $\vartheta(\psi) = p/\rho^k$ for adiabatic exponent k. However, to compel (1) to have fixed characteristics the author uses a four-parameter family of approximations

$$\lambda = w_1(\sigma) + w_2(\sigma)\tilde{\vartheta}_*(\psi),$$

where $w_{1\sigma\sigma}/w_1 = w_{2\sigma\sigma}/w_2 = m^2 = \text{constant}$, and $\tilde{\vartheta}_*(\psi)$ is at present arbitrary. Then (1) reduces to a type of equation solved by S. V. Vallander in a form depending in the present case on a solution of (2) $v_{\sigma\sigma} - m^2 v_{\theta\theta} + 2m\,\mathrm{cth}\,(m\sigma+a)v_\sigma = 0$ for constant a. When $\mathrm{cth}\,(m\sigma+a)$ is approximated by $1/(m\sigma+a)$ (2) reduces to an Euler-Darboux equation, and Vallander's solution yields an approximation to ψ in terms of three arbitrary functions and four parameters. The map from the (σ, θ)-plane to the (x, y)-plane is given by quadratures. *J. H. Giese* (Havre de Grace, Md.).

Nickel, Karl. Der höchstmögliche Auftrieb von Tragflügeln. Z. Angew. Math. Mech. **34**, 374–385 (1954). (English, French and Russian summaries)

Let the lift distribution along the span of a wing be denoted by $\gamma(y)$, $a \leqq y \leqq b$, and assume that the distributions of maximum and minimum lift, corresponding, respectively, to the occurrence of maximum and of minimum lift coefficient at every section, are known: $c(y) \leqq \gamma(y) \leqq C(y)$. In this paper the distribution $\gamma(y)$ for maximum total lift is then determined under an auxiliary condition representing specified rolling moment, pitching moment (of swept wing), or the like. *W. R. Sears* (Ithaca, N. Y.).

***Timman, R. La théorie des profils minces en écoulement non stationnaire en fluide incompressible ou compressible.** Journées de Mécanique des Fluides, Marseille 1952, pp. 285–327. Publ. Sci. Tech. Ministère de l'Air Paris, no. 296 (1951).

The various methods of solution for the problem of the title are reviewed and compared. No new results are presented, but the paper provides a convenient exposition for students or new workers in the field. *J. W. Miles.*

Legras, J. La seconde approximation de l'aile élancée en écoulement subsonique. Rech. Aéro. no. **42**, 17–21 (1954).

In the case of a slender wing, such as a delta wing, the integral equation for the circulation distribution $\Gamma(x, y)$ can be written

$$W_0(x, y) = -V\alpha(x, y) + W_1(x, y),$$

where V is the flight speed, $\alpha(x, y)$ is the local slope of the surface, and $W_0(x, y)$ is the upwash induced far downstream by the trailing vortices. $W_1(x, y)$ denotes the remainder of the upwash at any point and is given by a surface integral involving the circulation distribution $\Gamma(x, y)$. The Munk-Jones approximation is $W_1 = 0$; i.e., the boundary condition is satisfied by the trailing vortices. Here a second approximation is proposed in which the Munk-Jones results, say $\Gamma_0(x, y)$, is used to calculate W_1, at least after some numerical approximations to its kernel. For triangular wings, for example, an explicit result can then be obtained for W_1, but it is logarithmically infinite at the trailing edge. [This is presumably the same phenomenon encountered by Adams and Sears, J. Aero. Sci. **20**, 85–98 (1953); MR **14**, 599].

The author therefore applies Lighthill's technique (perturbation of independent variable) to remove this singularity in the second approximation. The effort is apparently successful. The equation for the coordinate shift is solved numerically. The resulting circulation distribution has the form

$$\Gamma(x, y) = K(x)(1 - \epsilon\Delta\alpha)\Gamma_0(x, y),$$

where $K(x)$ is a correction factor which removes the logarithmic singularity and $\epsilon\Delta\alpha$ denotes regular second-order terms. *W. R. Sears* (Ithaca, N. Y.).

Clippinger, R. F. Supersonic axially symmetric nozzles. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. No. **794**, 38 pp. (1951).

Calculations of isentropic steady flow in round nozzles have been carried out by the method of characteristics on the ENIAC machine. The theory is derived here and the approximation by difference equations is explained in detail. Results are given for twenty nozzles: four at each of five

exit Mach numbers. Results are available at the Ballistic Research Laboratories for ten nozzles, of various lengths, at each of 21 exit Mach numbers ranging from 1.008 to 8.238. The tables given here are samples. The nozzles are designed for a plane sonic surface at inlet and for uniform outlet conditions. *W. R. Sears* (Ithaca, N. Y.).

Nocilla, Silvio. Sul problema dell'ala triangolare a diedro qualsiasi a velocità supersonica, con incidenza e deriva piccole del prim'ordine. Aerotecnica **34**, 126–141 (1954).

This paper is concerned with the aerodynamic forces which act on a yawed delta wing at incidence at supersonic speeds. The delta wing is supposed to have an angle of dihedral β which (unlike the incidence and the angle of yaw) need not be infinitesimally small. The leading edges of the wing are supposed to be subsonic and the analysis is based on the linearised equations of supersonic flow. Cone-field theory is used in the early stages and the problem is then reduced to the solution of a singular integral equation of the first kind. The quantities involved are expanded in powers of the angle of dihedral β and the solution is determined as far as the third power of β. The functions which arise are investigated in detail and it is shown that the first two terms of the expansion agree with known solutions of the problem for infinitesimal β. Numerical results are included. *A. Robinson* (Toronto, Ont.).

von Baranoff, Alexis. Sur la résistance d'un corps de révolution effilé en mouvement accéléré ou décéléré. C. R. Acad. Sci. Paris **240**, 591–593 (1955).

The reviewer's published results on transonic drag [J. Aero. Sci. **21**, 644–645 (1954); MR **16**, 194] are extended over a wider range of Mach number. The algebraic results are in agreement with unpublished results of the reviewer and L. Wong, but the numerical results are not. In particular, the author obtains numerical values of the transonic drag in excess of the limiting, supersonic value, whereas the reviewer's results (loc. cit. ante) prove this to be impossible. *J. W. Miles* (Los Angeles, Calif.).

Thrane, P. Some hydrodynamical properties of simple atmospheric oscillations with applications to the semi-diurnal oscillation. Geofys. Publ. Norske Vid.-Akad. Oslo **18**, no. 1, 36 pp. (1951).

This paper deals with atmospheric oscillations which resemble tidal waves but are created by periodic supply and removal of heat, the gravitational effects of the sun and moon being neglected. The model is assumed to be frictionless and the basic flow is taken to be a zonal current which varies with height and latitude. The oscillations are regarded as small perturbations superposed on this flow so that the linearised hydrodynamic equations of motion can be used. A relation between the variation in the heat supply and the resulting pressure wave is deduced. For a barotropic model atmosphere an examination is made of the properties of oscillations having either a 12 hr. or a 24 hr. period. An expansion process is used to take account, to some extent, of the effect of the nonlinear terms and it is found that their presence admits the possibility of a slow, nonperiodic motion superposed on the oscillation. The variation of the supplied heat and of the vertical velocity is obtained by using observed values of the pressure wave, and the possibility of explaining this wave as a nonresonant oscillation resulting from thermal processes is discussed. *M. H. Rogers.*

Eliassen, A., Høiland, E., and Riis, E. Two-dimensional perturbation of a flow with constant shear of a stratified fluid. Institute for Weather and Climate Research, the Norwegian Academy of Sciences and Letters, Publ. No. **1**, 30 pp. (1953).

Riis, Eyvind. On perturbation of a linear flow with parabolic velocity profile. Vitenskaps-Akademiets Institutt for Vaer-og Klimaforskning, Oslo, Rap. no. **2**, i+13 pp. (1954).

Høiland, Einar. On the dynamic effect of variation in density on two-dimensional perturbations of flow with constant shear. Geofys. Publ. Norske Vid.-Akad. Oslo **18**, no. 10, 12 pp. (1953).

The paper by Eliassen, Høiland and Riis sets out to solve the following problem: an inviscid fluid, subject to gravity, flows in a fixed direction between two parallel, rigid horizontal planes at a vertical distance H apart. In this basic flow the velocity is $(U, 0, 0)$, where $U=\alpha z$, z being the vertical coordinate, and the density is proportional to $e^{-\beta z}$. This motion is then perturbed so that the velocity becomes $(U+u, 0, w)$, where u, w, and the perturbations in the density and pressure, are all functions of (x, z, t) alone. Both the basic and the perturbed flows are incompressible, by which is meant that the density is not a constant, but satisfies $d\rho/dt=0$. Thus a stream-function ψ, where $u=\partial\psi/\partial z$, $w=-\partial\psi/\partial x$, can be introduced in terms of which the non-zero component of the perturbation-vorticity is

$$\omega=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi=\nabla^2\psi.$$

It is then stated that the equation for $d\omega/dt$ is reducible to the following equation for ψ:

$$\left(\frac{\partial}{\partial t}+\alpha z\frac{\partial}{\partial x}\right)\nabla^2\psi+\beta g\frac{\partial^2\psi}{\partial x^2}=0. \tag{1}$$

If now $\psi=Z(z)\exp\{ik(x-\zeta t)\}$, it follows from (1) that Z satisfies Bessel's equation. The boundary conditions employed are (i) that $w=0$ at the rigid planes; (ii) that ω and $d\omega/dt$ at $t=0$ are given functions of z alone (not of x). The remainder of the paper consists of detailed investigations of the possible eigensolutions and of the general solution of this problem for the various Bessel functions and for different values of the parameter $\beta g/\alpha^2$. The aim is to discover which oscillations are stable and which unstable.

The reviewers have recalculated from first principles the equation for $d\omega/dt$ and have found that in (1) certain terms have been omitted. When these are included, the equation for Z is no longer Bessel's equation but one of more complicated type. To reduce the latter to Bessel's equation it is necessary to assume either (a) that H is negligibly small compared with β^{-1}, so that the unperturbed density must be effectively constant between the boundaries, or (b) the wave-length parameter k^{-1} must be negligibly small compared with the parameter β^{-1}.

The paper by Riis contains a discussion of a similar problem in which the basic flow is $U=U_0(1-4z^2/H^2)$ and the equation for $d\omega/dt$ is now stated to be

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\psi-\frac{d^2U}{dz^2}\frac{\partial\psi}{\partial x}=0. \tag{2}$$

But again (2) has been found to hold only if both the unperturbed and the perturbed densities are constants and not merely if $d\rho/dt=0$. Thus terms included by Eliassen,

Høiland and Riis are now excluded with the remark that "second-order terms are neglected", though no proof of their smallness is given. The rest of the paper is again devoted to considering the nature of the solutions for $k=0$ (infinite wave-length) and $k\neq 0$, particularly with respect to the stability or unstability of the motions.

The paper by Høiland treats the problem in which the basic flow is $U=\alpha z$, the density is proportional to $e^{-\beta z}$ and the wave-length of the disturbance is infinite. Equation (1) is taken as the equation governing the flow and two cases are discussed; firstly, flow in a channel bounded above and below by rigid planes; and, secondly, flow in a channel in which the upper boundary is a free surface.

G. C. McVittie and M. H. Rogers (Urbana, Ill.).

***Höiland, Einar. A mathematical problem from hydrodynamic perturbation theory.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 101–104 (1954). 25 Swedish crowns (may be ordered from Lunds Univeristets Matematiska Institution). (Norwegian)

A summary in Norwegian of the investigation contained in the preceding review. *G. C. McVittie.*

Høiland, Einar. On two-dimensional perturbation of linear flow. Geofys. Publ. Norske Vid.-Akad. Oslo **18**, no. 9, 12 pp. (1953).

This is an investigation of the stability of an inviscid incompressible fluid having a continuous vorticity distribution; the method of small perturbations is used and the equations are accordingly linearised. The author first treats the case of permanent waves, that is motions in which the amplitude remains constant, and finds a lower limit for the wavelength of such disturbances for two particular basic velocity profiles. By considering the case of self-excited and damped oscillations it is then proved that the flow of an inviscid fluid having a velocity profile of the form $U=A\cos\gamma z$ and bounded by two parallel walls $z=0$ and $z=h$ less than half a profile wavelength apart, is stable for all perturbations. Some general results on the stability of such flows are derived and in particular an upper limit for the amplification factor of a growing disturbance is deduced; this upper limit decreases with increasing values of the profile wavelength.

M. H. Rogers (Urbana, Ill.).

Kaspar'yanc, A. A. On the propagation of sound waves in a viscous gas in the presence of heat conduction. Prikl. Mat. Meh. **18**, 729–734 (1954). (Russian)

The author obtains general solutions of linearized equations for viscous, thermally conducting perfect gases, assuming the velocity, condensation, etc. are proportional to $e^{\sigma t}$, where σ is constant and the factors of proportionality are functions of position. These are given in terms of solutions of equations of the form $\Delta\varphi+k^2\varphi=0$, where k is a constant. It is unfortunate that he assumes the Stokes relation $3\lambda+2\mu=0$ connecting the viscosity coefficients since experimental evidence indicates that, for most gases, this relation is not satisfied. *J. L. Ericksen.*

***Kaspar'yantz, A. A. On the propagation of sound waves in a viscous gas with heat conduction.** Morris D. Friedman, Russian Translation, Two Pine Street, West Concord, Mass., 1954. 9 pp. (mimeographed) $4.50.

Translation of the paper reviewed above.

Elasticity, Plasticity

Ericksen, Jerald Laverne. Deformations possible in every isotropic, incompressible, perfectly elastic body. Z. Angew. Math. Phys. **5**, 466–489 (1954).

In this paper the following important problem of the theory of incompressible, isotropic elastic materials with strain energy is attacked: Assuming the absence of body forces, determine all deformations which are possible for every choice of the strain energy function. The author succeeds in solving this problem except for two rather special cases. He obtains all of the known exact solutions and discovers some new ones. One new solution is the deformation of a portion of a right circular cylinder into an oblique parallelepiped and another the deformation of a rectangular parallelepiped into a portion of a right circular cylinder bounded by two planes and two helical surfaces.

W. Noll (Berlin).

Biot, M. A. Theory of elasticity and consolidation for a porous anisotropic solid. J. Appl. Phys. **26**, 182–185 (1955).

The author proposes a theory purporting to describe the behavior of a porous elastic solid, the interconnecting pores of which are filled with a compressible fluid. The stress for the combined medium is assumed to be derivable from a strain energy V which is a quadratic function of two infinitesimal strain measures, calculated respectively from the "average" displacements of the solid and of the fluid. Only the trace of the latter occurs in V. The total stress is required to satisfy the equations of static equilibrium, which gives three equations involving the six displacement components. Three more equations are given which demand that the gradient of a hydrostatic part of the stress tensor be, to within a term proportional to the body force, a linear combination of the components of the velocity of the fluid relative to that of the solid. Simplified forms of these equations are given for the case where the combined medium is isotropic or transversely isotropic. *J. L. Ericksen.*

Hu, Hai-Chang. On the general theory of elasticity for a spherically isotropic medium. Sci. Sinica **3**, 247–260 (1954).

This paper contains a general solution, which is shown to be complete, of the equations of linear elasticity for a medium which is spherically isotropic. Such a medium is locally transversely isotropic, the axis of anisotropy varying with position in such a way as to remain normal to a set of concentric spheres. The solution is expressed in terms of solutions to two rather complicated differential equations or, alternatively, in power series in spherical harmonics. It is shown that a general solution for transversely isotropic materials previously studied by the author [Acta Sci. Sinica **2**, 145–151 (1953); MR **15**, 1004] results by a limit process.

J. L. Ericksen (Washington, D. C.).

Salzmann, Fritz. Wärmespannungen und -deformationen im elastischen Körper bei ebener stationärer Wärmeströmung. Z. Angew. Math. Phys. **3**, 129–148 (1952).

The general thermo-elastic equations are presented and specialised for plane strain. For steady heat flow in the plane without sources and sinks it is shown that stresses in the plane can be zero. The displacement expressed as a complex variable is represented by an integral. With sources and sinks, zero stresses in the plane are still possible if a slit

passes through all of them, and relative displacements across the slit are simply represented. Approximate solution for cylinder with free ends using St. Venant's principle is given by equalizing the stress normal to the plane. Stresses in a cylinder with hole with internal source are given exactly for a circular annulus and approximately otherwise. The special influence of the symmetry of the temperature distribution is discussed. Quite similar, and in some respects more revealing analysis, is given in Chap. 14, Theory of Elasticity by Timoshenko and Goodier [McGraw-Hill, New York, 1951; MR 13, 599], ascribing the ideas to earlier papers.

E. H. Lee (Providence, R. I.).

Natanzon, V. Ya. On a variation of the boundary condition of the plane problem of the theory of elasticity. Dokl. Akad. Nauk SSSR (N.S.) **98**, 27–29 (1954). (Russian)

The author considers the first fundamental problem of plane elasticity for simply connected regions with the parametric representation of the boundaries in the form

$$x=x(\xi), \quad y=y(\xi) \quad (0\leqq\xi\leqq 2\pi).$$

By replacing ξ by $u=\xi+i\eta$, one obtains a transformation of the original region into the upper semi-infinite strip $\eta\geqq 0$, $0\leqq\xi\leqq 2\pi$. Due to the multi-valued character of the transformation, the actual region mapped onto the strip has a cut, so that this method can be used for the study of regions containing a single cut. The boundary condition for this type of transformation is deduced and the method of power series applied to solve one particular case.

J. R. M. Radok (Melbourne).

Sokołowski, Marek. On certain two-dimensional problems concerning the theory of elasticity of orthotropic bodies. Arch. Mech. Stos. **6**, 65–92 (1954). (Polish. Russian and English summaries)

The author solves by means of Fourier integrals problems of plane elasticity for orthotropic media when the boundary involves straight lines. The particular cases considered are the rigid punch and the elastic beam acting on the boundary of the half-plane; the concentrated force acting anywhere in the half- or complete plane. *J. R. M. Radok.*

Sonntag, Rudolf. Die Methode der konformen Abbildung ebener Spannungszustände von L. Föppl, angewandt auf Probleme des Parallelstreifens. Z. Angew. Math. Mech. **34**, 435–438 (1954).

Following L. Föppl [same Z. **11**, 81–92 (1931)] the author uses conformal mapping to deduce L. N. G. Filon's [Philos. Trans. Roy. Soc. London. Ser. A. **201**, 63–155 (1903), p. 106] solution for the infinite strip, subject to two concentrated edge forces at opposite points of the boundary, from the known solution for the circle under concentrated forces. The obtained solution contains a Fourier integral, which converges better and hence is more suitable for computation than Filon's. The case of bending of beams by concentrated loads is also deduced from this solution.

J. R. M. Radok (Melbourne).

***Galin, L. A. Kontaktnye zadači teorii uprugosti. [Contact problems of the theory of elasticity.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 264 pp. 8 rubles.

The author deals exhaustively with the theoretical aspects of the problems arising from contact of two bodies at least one of which is elastic. A survey of work done on such problems inside and outside Russia is included.

There are two main parts, the first dealing with the plane, the second with the half-space problem. All basic formulae are deduced from first principles and a summary of the relevant work on Cauchy integrals, taken over infinite lines, on the solution of the Riemann-Hilbert problem, and on three-dimensional potential theory is given in order to make the book self-sufficient.

The particular cases considered comprise various shapes of rigid and elastic stamps, with or without friction and cohesion, stationary or moving, on a half-plane of isotropic or anisotropic material; circular, elliptic, triangular cylinders with or without friction pressing on the half-space. One typical general result is that the pressure distribution under a rigid stamp is the same for orthotropic as for isotropic half-planes, provided the boundary is parallel to one of the axes of orthotropy.

The presentation is clear and detailed and the book illustrates well the use of function theory in elasticity.

J. R. M. Radok (Melbourne).

Mossakovskiĭ, V. I. Application of a reciprocity theorem to the determination of resultant forces and moments in spatial contact problems. Prikl. Mat. Meh. **17**, 477–482 (1953). (Russian)

A die with surface $z=f(x, y)$ is pressed into an elastic half-space. When the pressure-distribution under a plane die of the same form is known, the finding of the resultant pressure and its moments on a die with arbitrary surface reduces to integrations. *W. H. Muller* (Amsterdam).

Chien, Wei-Zang. Assumptions in Saint-Venant's solution for the torsion of an elastic cylinder. Acta Sci. Sinica **3**, 165–170 (1954).

The author shows that St. Venant's solution for torsion of cylinders follows from the assumptions that

$$\partial\tau_{zz}/\partial z=\partial\tau_{yz}/\partial z=0,$$

that the sides of the cylinder are free of tractions, and that the normal stress σ_z acting on a plane end is self equilibrated. References to equivalent assumptions discovered by other authors are included. *J. L. Ericksen.*

Suharevskiĭ, I. V. On the problem of torsion of a composite multiconnected bar. Inžen. Sb. **19**, 107–124 (1954). (Russian)

The Saint-Venant torsion problem for a compound beam (=a cylinder reinforced by the longitudinal rods) was first investigated by N. I. Muskhelišvili [Some basic problems of the mathematical theory of elasticity, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, §139, 140; MR **11**, 626; **15**, 370] who reduced it to the solution of an integral equation for an auxiliary function. The stresses in the beam are then determined by differentiating this function. In this paper an integral equation (2.27) is deduced which yields directly the distribution of stresses on the boundaries. The equation is solvable in Neumann's series. The paper is illustrated by an example relating to a beam with a triply-connected cross section formed by three circles. The concluding paragraph contains an integral equation (7.1) for the shearing stresses on the boundaries of a composite beam whose cross-section consists of n regions S_i, $i=1, \cdots, n$, such that each S_i is wholly contained within S_{i+1}.

I. S. Sokolnikoff (Los Angeles, Calif.).

Šerman, D. I. Bending by a transverse force of an elliptic beam weakened by a longitudinal circular cylindrical cavity. Inžen. Sb. **17**, 121–150 (1953). (Russian)

The author uses N. I. Muskhelishvili's approach to set up the problem of determination of the flexure function for a doubly connected cross-section bounded by an ellipse and a circle [Some basic problems of the mathematical theory of elasticity, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, Chap. 22; MR **11**, 626; **15**, 370]. Having solved the problem by function-theoretical methods, he discusses several numerical cases to re-assess the validity of Jouravski's formula [Todhunter and Pearson, A history of the theory of elasticity, vol. II, part 1, Cambridge, 1893, §939].

J. R. M. Radok (Melbourne).

Arutyunyan, N. H., and Gulkanyan, N. O. On the center of bending of certain prismatic bars with polygonal cross-section. Prikl. Mat. Meh. **18**, 597–618 (1954). (Russian)

The authors deduce exact solutions of St. Venant's problem for bars with certain polygonal cross-sections with one axis of symmetry. The solutions are obtained by considering separately rectangular subregions of the cross-sections and introducing auxiliary harmonic functions which permit the subsequent joining up of the solutions in the subregions. This process leads to systems of second-order ordinary differential equations for the variable coefficients of the Fourier expansions of the component functions of the solutions. The determination of the integration constants finally leads to infinite systems of simultaneous linear equations which arise from the joining up of the solutions in the subregions. Particular cases considered are the cross-sections **T**, **U**, **V** with right angles at the corners.

J. R. M. Radok (Melbourne).

Radenković, D. Bending of a curved bar in its own plane. Quart. J. Mech. Appl. Math. **7**, 385–398 (1954).

The present paper deals with the problem of bending of bars from the point of view of a linearized theory founded upon the assumptions that (a) the influence of the deflection on the conditions of equilibrium has to be taken into account, (b) the deflection is, however, such that the squares and the products of the displacements and their derivatives can be neglected. The author also assumes that the material is perfectly elastic and that the central line of the bar lies in a plane which is also the plane of bending of the bar. Linearizing the equations of equilibrium of G. Kirchhoff and A. Clebsch and introducing a suitable fictitious loading, the basic integral equations of the problem are formulated. These equations permit a qualitative analysis of the problem and at the same time yield a ready numerical analysis of any particular case. The application and the practical importance of the proposed theory are shown in a numerical example of the analysis of a bridge arch.

R. Gran Olsson (Trondheim).

***Gol'denveĭzer, A. L. Teoriya uprugih tonkih obolочek. [Theory of thin elastic shells.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 544 pp. 23.50 rubles.

This monograph contains the most comprehensive treatment of the theory of thin, isotropic elastic shells thus far published. To a large extent, it is based on the original investigations of the basic relations of the shell theory, phrased as a system of differential equations in which the thickness of the shell appears as a parameter. This approach has enabled the author to deduce the diverse approximate methods of analysis of shells and to characterize the range of applicability (including useful estimates of errors) from a unified point of view.

The exposition is clear, though occasionally repetitious, presumably to preclude the reader from irksome leafing-back. The notation is patterned after Love, but the treatment depends on liberal use of vector analysis. The formulation is given in curvilinear coordinates, in the most general form, without the benefit of tensor calculus. The mathematical rigor is sufficient unto the theory.

The book is made up of five fairly independent parts. Part I (103 pp.) develops the basic formulas and theorems of the general linear theory of shells. It is divided into four chapters containing: a) a summary of basic equations of geometry of surfaces, b) an account of static and geometric relations of the shell theory, c) the stress-deformation relations and energy theorems, and d) a derivation of complete systems of equations based on the hypothesis of non-extensibility of elements of normals to the middle surface.

Part II (101 pp.) is given entirely to the membrane, or momentless, theory of shells. It consists of five chapters treating: a) the theory of arbitrary shells, b) the theory of shells of zero curvature, c) the theory of spherical shells, d) the stress analysis of closed spherical shells, and e) the stress analysis of open shells. In this part, and throughout the book, the membrane theory is treated as a special case of the more general bending, or moment, theory in which the stress couples produced by bending are not neglected. The main result here is a general integral of equations of the membrane theory, which serves, in Part V, to characterize the range of validity of diverse approximations.

Part III (78 pp.) is devoted to a study of circular cylindrical shells with the aid of the trigonometric series. Several known approximate methods of analysis of cylindrical shells are deduced by specializing the exact formulas. The treatment covers both closed and open shells and includes a careful study of the edge effect.

Part IV (119 pp.) deals with the analysis of the stressed states of arbitrary shells. This analysis is preceded by a study of a purely mathematical problem concerned with the asymptotic integration of a partial differential equation with vanishingly small coefficients in the higher-order derivatives. This equation is a prototype of equations of the shell theory. The asymptotic integration of equations in the shell theory is a problem of considerable mathematical complexity and delicacy, and the author is obliged to introduce a number of reasonable assumptions in lieu of proofs. The matters of convergence of processes of asymptotic integration are set aside.

Part V (141 pp.) treats several approximate methods of analysis. The point of departure in the study of approximate methods is, naturally, the membrane theory. A set of sufficient conditions and criteria for the applicability of this theory is formulated. It is then supposed that one or the other of the stipulations of the membrane theory is violated and a special approximate method is constructed for problems in the appropriate category. Each of such methods is provided with an estimate of the magnitudes of errors. Special attention is given to conical and cylindrical shells. The author justifies the omission of detailed analyses of shallow shells and shells of revolution by a statement that such shells have been adequately treated in the monographs by V. Z. Vlasov [The general theory of shells, Gostehizdat,

Moscow-Leningrad, 1949; MR 11, 627], V. V. Novozhilov [Theory of thin shells, Gostehizdat, Moscow, 1951] and A. I. Lur'e [Statics of thin elastic shells, Gostehizdat, Moscow, 1947; MR 12, 301]. The reviewer had no access to these monographs.

The volume deserves the considered attention of all serious workers in the shell theory and is quite without parallel in the western literature. *I. S. Sokolnikoff.*

Ambarcumyan, S. A. On the limits of applicability of certain hypotheses of the theory of thin shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 5, 57–72 (1954). (Russian)

The author considers equations of equilibrium for thin shells similar to those given, e.g., in Love [A treatise on the mathematical theory of elasticity, 4th ed., Cambridge, 1927] and equations which result from these if one regards certain resultant couples as negligible, three different approximations being discussed. He attempts to delimit the range of validity of the approximate equations by comparing exact solutions of the exact and approximate equations. For a shell having the form of a part of a right circular cylinder, he obtains general solutions of the equations considered in the form of trigonometric series. For particular types of loading, the author gives tables indicating the error resulting from using the approximate equations for shells of different dimensions. This reviewer was unable to draw any general conclusions from this work. *J. L. Ericksen.*

Darevskiĭ, V. M. On the theory of cylindrical shells. Prikl. Mat. Meh. **15**, 531–562 (1951). (Russian)

The author studies the solution of the general equations of equilibrium of cylindrical shells subject to external loading which is uniform over an area bounded by four segments of lines of principal curvature (elementary loading). The series solutions are discussed and their convergence is proved with a view to using these elementary loading cases to build up more complicated loadings by a limit process. In particular, the case of forces or moments, concentrated at points or along lines of principal curvature are treated fully. *J. R. M. Radok* (Melbourne).

Darevskiĭ, V. M. Solution of certain questions of the theory of a cylindrical shell. Prikl. Mat. Meh. **16**, 159–194 (1952). (Russian)

This paper continues the paper reviewed above and studies in greater detail the limiting process introduced there. Special consideration is given to the deduction of asymptotic expansions of the solutions in the neighbourhood of the regions of application of the external loadings.

J. R. M. Radok (Melbourne).

Zenova, E. F., and Novožilov, V. V. Symmetric deformation of a toroidal shell. Prikl. Mat. Meh. **15**, 521–530 (1951). (Russian)

The authors study the relevant differential equation for values of a parameter for which earlier power-series solutions were very slowly converging and which nevertheless were of practical interest. They find approximate solutions in terms of Hankel functions whose power series and asymptotic expansions have more satisfactory convergence properties.

J. R. M. Radok (Melbourne).

Nowiński, J. Some problems of the theory of thin-walled tubes. Arch. Mech. Stos. **4** (1952), 123–163 (1953). (Polish. English summary)

This paper is a summary of the author's following previous publications on the subject: Prace Głown. Inst. Lotn. **1952**, no. 1; Technika Lotnicza **5**, no. 1(8), 2–8 (1950); **6**, 2–13 (1951); Arch. Méc. Appl., Gdańsk **3**, 53–60 (1951); MR **13**, 405.

The author discusses the development of the theory of thin-walled beams, states the basic assumptions, introduces his frame of reference, derives the equations of equilibrium, shows solutions, discusses the center of shear and twist and gives a long list of references. A rather similar theory was presented also by J. Hadji-Argyris and P. C. Dunne [J. Roy. Aero. Soc. **51**, 199–269, 757–784, 884–930 (1947); **53**, 461–483, 558–620 (1949); MR **8**, 613; **9**, 122, 256; **10**, 651]. *T. Leser* (Lexington, Ky.).

Storchi, Edoardo. Sulle membrane aventi comportamento meccanico eccezionale. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 462–483 (1953).

This is apparently the final paper in a sequence of studies of the general solution of the equations of equilibrium for a two-dimensional real Riemannian space [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 227–231; **8**, 116–120, 326–331 (1950); Riv. Mat. Univ. Parma **3**, 339–360 (1952); MR **11**, 556, 757; **12**, 219; **14**, 1035]. The author has shown earlier that for the general surface, the general solution is an expression in the partial derivatives of orders 1 through 5 of a scalar function. He now characterizes the exceptional surfaces for which no fifth derivatives are necessary. He shows that (1) second derivatives are enough if and only if the Gaussian curvature K is constant, (2) third derivatives are enough if and only if K does not vary along the geodesic meridians issuing from a pole, (3) fourth derivatives are enough if and only if the ratio of the rates of change of K along the geodesic meridians and parallels does not vary in the directions of the parallels. He characterizes these special types of surface by properties of applicability.

C. Truesdell (Bloomington, Ind.).

Rüdiger, D. Der Spannungs- und Verschiebungszustand drehsymmetrischer Membrane mit beliebig gekrümmter Meridiankurve. Ing.-Arch. **22**, 336–347 (1954).

The author treats axisymmetric membranes. The problem is resolved into its Fourier components $\cos n\psi$ and $\sin n\psi$, where ψ is the azimuthal coordinate. For $n=0$ and 1, the stress and strain distributions can be found for all shapes of membrane. For $n \geqq 2$, the solution is found for only two shapes. For negatively curved membranes, interesting special cases arise where no stress solution satisfies this theory and the strain is indeterminate. *D. R. Bland.*

Levi, Beppo. Essay on the calculation of the deflection of thin plates. Math. Notae **12–13**, 79–193 (1954). (Spanish)

A particular solution of the equation

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2 \partial y^2}+\frac{\partial^4 u}{\partial y^4}=f(x, y)$$

is obtained by two different computations. The first uses trigonometrical developments the second polynomials.

G. Fichera (Trieste).

Winslow, A. M. Differentiation of Fourier series in stress solutions for rectangular plates. Quart. J. Mech. Appl. Math. **4**, 449–460 (1951).

Certain mathematical conditions of differentiation are necessarily included in a Fourier solution which is based on the hypothesis that a stress function and its partial derivatives can be represented by Fourier series. To illustrate these conditions, a summary of essential theory is followed by examples, using full-range Fourier series and also half-range Fourier series.

In these examples of stress solutions it is found that the assigned boundary tractions provide equations sufficient to determine the Fourier coefficients. Additional mathematical conditions, however, are imposed by the hypothesis of termwise differentiation. Unless it can be proved that these additional conditions are fulfilled, the conclusion is that the hypothetical representation by Fourier series may not have sufficient generality to satisfy all required conditions and furnish a solution. (Author's summary.) *R. M. Morris.*

Szelągowski, Franciszek. The influence of a bolt driven into a hole in a plate subjected to tension or bending. Arch. Mech. Stos. **6**, 365–388 (1954). (Polish. Russian and English summaries)

Complex function theory is used to solve particular cases of circular holes in infinite plates with inserted, larger diameter, elastic discs. [See N. I. Muskhelishvili, Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR 11, 626; 15, 370.] *J. R. M. Radok* (Melbourne).

***Mettler, E. Le problème des oscillations non linéaires des corps élastiques. Zum Problem der nicht-linearen Schwingungen elastischer Körper.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 77–96, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and German)

The author is concerned with nonlinear vibration problems in elasticity, particular attention being paid to the case where the applied forces per unit undeformed area are of the form $p_i^1+p_i^2\epsilon\cos\omega t$, where ϵ and ω are constants and p_i^1 and p_i^2 are functions of initial position, but not of the time. For the case where the equilibrium displacement u_i^1 corresponding to the load p_i^1 is given with sufficient accuracy by linear elasticity, the author gives a variational method for determining second-order approximations to the total displacement. There are, in general, an infinite number of approximate solutions consistent with the given boundary data. It may happen that, for (ϵ, ω) in some regions of the (ϵ, ω)-plane, some of the approximate displacements increase without bound with increasing time. If one of these regions includes a point for which $\epsilon=0$, the equilibrium state is regarded as unstable. Stability or instability is determined by studying the stability of a coupled system of second order, ordinary differential equations obtained from the variational procedure.

Other possibilities, such as permitting the equilibrium displacement to be that given by a nonlinear theory, are briefly discussed, and a fairly extensive bibliography is given. *J. L. Ericksen* (Washington, D. C.).

Morse, R. W. Compressional waves along an anisotropic circular cylinder having hexagonal symmetry. J. Acoust. Soc. Amer. **26**, 1018–1021 (1954).

Exact solutions are derived for the propagation of compressional elastic waves along an anisotropic circular cylinder having axial elastic symmetry (hexagonal elastic symmetry with the crystallographic axis coincident with the cylinder axis). Formal solutions are obtained for the displacement components, and a transcendental equation relating the phase velocity and the circumference-to-wavelength ratio is derived. The solutions are shown to reduce to the Pochhammer solution for the isotropic case. In the long-wavelength extreme the velocity is dispersionless and is given in terms of the axial Young's modulus. In the short-wavelength limit the velocity again becomes constant, similar to the Rayleigh surface-wave limit in the isotropic case. (Author's summary.) *R. M. Morris* (Cardiff).

Budiansky, Bernard, and Kruszewski, Edwin T. Transverse vibrations of hollow thin-walled cylindrical beams. NACA Tech. Note no. **2682**, 29 pp. (1952).

This problem is treated by means of Hamilton's principle. Transverse vibration parallel to an axis of symmetry of the cross-section is considered. The cross-section is assumed to retain its shape and the stresses to be uniform across the wall thickness. Fourier series expansions for the lateral displacement $w(x)$, where x is the beam length coordinate, for the longitudinal displacement $u(x, s)$ are taken, where s is the arc length along the perimeter of the section. This permits shear deformation of arbitrary distribution, which thus introduces both shear-deformation and shear-lag effects into the beam vibration theory. The kinetic energy associated with $\dot{u}$ is included, termed longitudinal inertia effect, and this is equivalent to the rotary inertia of the non-shear-lag problem. Numerical examples are computed for beams of rectangular section, and the influence of transverse shear, shear lag, and longitudinal inertia is discussed.

E. H. Lee (Providence, R. I.).

Matschinski, Matthias. Sur les vibrations d'une plaque (ou d'une couche) plane infinie et sur les mesures pour déterminer son épaisseur. C. R. Acad. Sci. Paris **239**, 1766–1768 (1954).

Bishop, R. E. D. Longitudinal waves in beams. Aero. Quart. **3**, 280–293 (1952).

A one-dimensional equation for longitudinal waves in rods is developed taking into account lateral inertia and shear stresses in the plane of the section. It is obtained by a strength-of-materials type argument, starting from the full equations of elasticity and simplifying them by integrations and plausible assumptions. The discussion is simple, revealing, and applies to non-circular convex sections. For short wavelengths an asymptotic wave velocity is approached as for the exact theory, and the predicted phase velocities show a marked improvement over the simplest theory and that corrected for lateral inertia by Rayleigh. Higher order approximations are developed in the same manner. A more analytical attack on this problem by Mindlin and Herrmann [Proc. 1st U. S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 187–191] which utilizes results of the exact theory provides a more accurate one-dimensional theory for circular rods. *E. H. Lee* (Providence, R. I.).

Pinney, Edmund. Surface motion due to a point source in a semi-infinite elastic medium. Bull. Seismol. Soc. America **44**, 571–596 (1954).

The author begins the discussion with the fundamental equations of the theory of elasticity; no references are made

to the work done on this problem during the last 50 years. For both kinds of sources the author considers solutions of the wave equations of a very special form, namely,

$$-(V_0/4\pi^2R)\{\tan^{-1}[\epsilon/(R/c-t)]+\tan^{-1}[\epsilon/(R/c+t)]\},$$

where R is a distance from a source, V_0 a constant, and ϵ is positive and small. This expression should hold for $t \geqq 0$. The author then transforms the expressions obtained to a form suitable for numerical calculations, the results of which are presented in several tables. *W. S. Jardetzky.*

Satô, Yasuo. Study on surface waves. XII. Non-dispersive surface waves. Bull. Earthquake Res. Inst. Tokyo **32**, 349–360 (1954). (Japanese summary)

***Korotkin, Ya. I., Lokšin, A. Z., i Sivers, N. L. Izgib i ustoĭčivost' steržneĭ i steržnevyh sistem. (Stroitel'naya mehanika korablya.) [Bending and stability of beams and beam systems. (Structural mechanics of ships.)]** Gosudarstv. Naučno-Tehn. Izdat. Mašinostroitel. i Sudostroitel. Lit., Moscow-Leningrad, 1953. 519 pp. 13.40 rubles.

In eight chapters the authors present the theory of structures, formulated with special consideration being given to ship design. Intended as a textbook for students and reference book for practising engineers, the book develops the various known methods for solving the differential equations referring to beams, grids of beams, frames, arches. Special consideration is given to beams on elastic foundations and to problems of stability. Numerical examples illustrate the methods throughout and most of the results pertaining to beams are summarized conveniently at the end of the book.

The chapter headings are: I) Application of the principle of virtual work to the study of the equilibrium of elastic bodies; II) Bending of straight beams; III) Plane frames, consisting of straight beams; IV) Curved frames; V) Bending of beams on elastic foundations; VI) Plane grids of beams; VII) Combined loading of beams; VIII) Stability of beams. *J. R. M. Radok* (Melbourne).

Voroncov, G. V. Generalization of some theorems of P. F. Papkovič on the stability of elastic systems. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 127–132 (1953). (Ukrainian. Russian summary)

The theorems of Papkovič and the author deal with the calculation of approximations for the critical loads and the natural frequencies of elastic systems. The maximal error of the approximation can be indicated. *W. H. Muller.*

Ambarcumyan, S. A. On computation of the stability of thin-walled rods. Akad. Nauk Armyan. SSR. Dokl. **17**, 9–14 (1953). (Russian. Armenian summary)

The author studies the stability of thin-walled cylindrical shells with discontinuous principal curvature and open cross-section, using the equilibrium equations.

J. R. M. Radok (Melbourne).

Buckens, F. Théorie limite du flambage d'une plaque circulaire chauffée en son centre. Déformées caractéristiques. Ann. Soc. Sci. Bruxelles. Sér. I. **68**, 157–163 (1954).

L'auteur a determiné précédemment [mêmes Ann. **68**, 63–71 (1954); MR **16**, 93] les valeurs critiques de la température du chauffage local au centre d'une plaque circulaire, valeurs pour lesquelles se produit le début de l'instabilité élastique sous l'action de la dilatation thermique. L'auteur continue à nous limiter aux déformées à symétrie circulaire, dans l'hypothèse où le rayon r_0 de la zone chauffée est très petit par rapport au rayon R de la plaque. Les conditions de contrainte à la périphérie (encastrement, appui simple) n'ont alors q'une faible influence sur les valeurs et les déformées caractéristiques. Le spectre des températures critiques, pour $|\log \rho|^{-1}=|\log r_0/R|^{-1} \ll 1$ est donné par la formule (1) $T_s=(h/r_0)^2\lambda_s/6\alpha(1-\nu^2)$ où l'on a posé: (2) $\lambda_s=1+s^2\pi^2/\log^2 \rho$ pour $s=1, 2, \cdots$; h est l'epaisseur, et α le coefficient de dilatation. La formule (2) est valable tant que $s^2\pi^2/\log^2 \rho \ll 1$. On voit facilement que lorsque le rayon de la plaque tend vers l'infini ($|\log \rho| \to \infty$), le spectre (1) devient continu à partir de la valeur minimum: $T_0=(h/r_0)^2/6\alpha(1-\nu^2)$. Les gradients des déformées caractéristiques, $\varphi=dw/dr$, sont donnés par une formule, qui se réduit, au voisinage du centre de la plaque,

$$\varphi \cong \sin\{s\pi \log r_0/r \log \rho\}$$

d'où:

$$w \cong \frac{r}{2r_0}\left\{\sin s\pi \log \frac{r_0}{r}\Big/\log \rho+\cos s\pi \log \frac{r_0}{r}\Big/\log \rho-(-1)^s\right\}$$

le déplacement transversal w diminuant rapidement à une certaine distance du centre de la plaque, lorsque ρ est très petit. *R. Gran Olsson* (Trondheim).

***Prager, W., und Hodge, P. G., Jr. Theorie ideal plastischer Körper.** Ins Deutsche übertragen von F. Chmelka. Springer-Verlag, Wien, 1954. x+274 pp. DM 33.00; $7.85.

Translation, with correction of known errors, of Theory of perfectly plastic solids [Wiley, New York, 1951; MR **14**, 430].

Hopkins, H. Geoffrey, and Prager, William. On the dynamics of plastic circular plates. Z. Angew. Math. Phys. **5**, 317–330 (1954).

The paper is concerned with the dynamics of a thin circular plate made of a plastic-rigid material that obeys the yield condition of Tresca and the associated flow rule. In the case of rotational symmetry the basic equations of the dynamic problem are established by making use of the physical concepts employed by the authors in a previous paper [J. Mech. Phys. Solids **2**, 1–13 (1953); MR **15**, 270]. The discontinuities which might occur in the mechanical quantities and in their derivatives with respect to time and to the radial coordinate are then investigated. Because of the rotational symmetry such discontinuities occur across the concentric circles. In the authors' analysis these circles replace the yield hinges of the corresponding analysis of plastic beams and frames.

In the last part of the paper the complete solution of the basic equations is given for a simply supported plate subjected to a uniformly distributed load which is brought on suddenly and after a certain time interval removed suddenly.

E. T. Onat (Ankara).

Wang, A. J., and Hopkins, H. G. On the plastic deformation of built-in circular plates under impulsive load. J. Mech. Phys. Solids **3**, 22–37 (1954).

The paper is concerned with the dynamics of a thin circular plate. The dynamic load is supposed to be such that at some given time a uniform transverse velocity is suddenly imparted to the entire plate, save at the edge where the velocity is zero and thereafter the plate is subject only to

edge forces which effect built-in conditions of support. By using the analysis formulated by Hopkins and Prager in the paper reviewed above, the authors determine the motion of the plate and obtain estimates of the time that elapses before the plate comes to rest and of the plate central deflection at that time. It is noted in the paper that a similar problem with simple support conditions was solved by Wang [Brown Univ. Tech. Rep. **DA-798-7** (1953)].

E. T. Onat (Ankara).

Hopkins, H. G., and Wang, A. J. Load-carrying capacities for circular plates of perfectly-plastic material with arbitrary yield condition. J. Mech. Phys. Solids **3**, 117–129 (1955).

A general theory is presented for determining the load carrying capacity of a symmetrically loaded circular plate made of a perfectly plastic material satisfying an arbitrary yield condition. It is shown that the problem can be reduced to the solution of a first-order ordinary differential equation. For the Tresca yield condition, previously treated by Hopkins and Prager [same J. **2**, 1–13 (1953); MR **15**, 270], this equation can be integrated exactly, but in general a numerical solution is required. This is done for a simply supported plate under a uniform load for both the Mises and the "parabolic" yield condition. Also treated are simply supported and built-in plates satisfying the Mises yield condition under a uniform load over a central portion only of the plate. The numerical solutions are carried out by the method of isoclines and also by an iterative procedure using the closed-form Tresca solution as a first approximation.

As the authors point out, their solution provides only a lower bound unless a velocity field can be associated with the given stress field by the plastic potential stress-strain law. However, they show that such a velocity field can be found by a double numerical quadrature. [Reviewers note: For the simply supported plate where these quadratures are explicitly exhibited, it is evident that the resulting velocity field will be satisfactory and hence there is no need to carry out the integrations. However, in extending the techniques to other examples, it is conceivable that the velocity field might not satisfy certain sign conditions to which the authors call attention.] *P. G. Hodge* (Brooklyn, N. Y.).

Onat, E. T., and Prager, W. Limit analysis of shells of revolution. I. Nederl. Akad. Wetensch. Proc. Ser. B. **57**, 534–548 (1954).

The stress distribution in a shell of revolution subjected to an axially symmetric load is most conveniently described in terms of the bending moments M_θ and M_ϕ and the membrane forces N_θ and N_ϕ. For a perfectly plastic material it is necessary to express the yield condition and flow rule in terms of these resultants and the corresponding generalized strain rates, κ_θ, κ_ϕ, ϵ_θ, ϵ_ϕ. Under the assumption that the shell material satisfies Tresca's yield condition, the authors do this by an ingenious extension of techniques used by Drucker [Proc. 1st Midwest Conference Solid Mech., Univ. of Ill., Urbana, Ill., 1953, pp. 158–163] and Onat [Quart. Appl. Math. **13**, 63–72 (1955)] for cylindrical shells. The basic idea is to consider possible strain patterns through the shell thickness of a fully plastic section, and to determine the corresponding stress resultants by means of the flow law. The result defines a closed convex surface in a four-dimensional stress resultant space.

The authors show that their result reduces to that of Onat if the shell is a cylinder. Finally, upper and lower bounds are found for the load-carrying capacity of built-in spherical caps, using the principles of limit analysis. These bounds become closer with decreasing thickness-to-radius and with increasing cap angle. *P. G. Hodge.*

Neményi, P. F., and van Tuyl, A. Two-dimensional plastic stress systems with isometric principal stress trajectories. Quart. J. Mech. Appl. Math. **5**, 1–11 (1952).

The totality of stress fields and corresponding yield conditions in two dimensions which determine isometric principal stress trajectories is studied. These are sought irrespective of whether the resulting yield condition approximates that for an actual material. Some known yield conditions are obtained, others are unknown and sometimes inapplicable to known materials, for example the "yield condition" that the average stress is a constant. In addition to the particular solutions, the general development of the theory loads to trivial nets which apply for any yield function. Some common yield functions exhibit only such nets. There is an introductory section in which the deduction of two-dimensional yield functions from three-dimensional ones is discussed. *E. H. Lee* (Providence, R. I.).

Anan'ina, A. N. Axially symmetric deformations of a cylindrical shell with elastic-plastic deformations. Inžen. Sb. **18**, 157–160 (1954). (Russian)

An approximate analysis is given of the deformation of an infinitely-long circular cylindrical shell subjected to a ring of concentrated load. The shell material is non-hardening plastic-elastic. The analysis proceeds through the assumption that at a shell normal cross-section there is either completely elastic or completely plastic mechanical behaviour. It is difficult to assess the exact consequences of this assumption. *H. G. Hopkins* (Fort Halstead).

Besseling, J. F. A theory of plastic flow for anisotropic hardening in plastic deformation of an initially isotropic material. Nationaal Luchtvaartlaboratorium, Amsterdam. Rep. **S. 410**, ii+30+iv (18 plates) (1953).

The author considers small deformations of a work-hardening material. The material is assumed to be composed of various components. Each component has a different yield point and it may also exhibit work-hardening. The deformation of an element of the material is then investigated by introducing certain assumptions and by making use of a kind of elastic potential. The stress-strain relations obtained in this manner are essentially of the flow type. But the yield surface might develop corners during the plastic loading of the material. The author points out that this type of stress-strain relations are more realistic. The Bauschinger effect could also be predicted by a theory of this type. [Similar refinements of the stress-strain relations have also been suggested by other authors: W. T. Koiter, Biezeno Anniversary Volume, Stam, Haarlem, 1953, p. 232–251; MR **14**, 1148; J. L. Sanders, Proc. Second U. S. Natl. Congr. on Appl. Mech., Ann Arbor, Mich., 1954 (in press); W. Prager, J. Appl. Mech. **20**, 317–320 (1953); MR **15**, 583].

In order to compare the theory with experiments, test results on torsion and also on the combined torsion and compression of thin walled tubes are given at the end of the paper. Good agreement is found between the theory and the experiments. However the author's method of determining the number and the specifications of the components seems to be rather arbitrary. *E. T. Onat* (Ankara).

Steele, M. C. The plastic bending and twisting of square section members. J. Mech. Phys. Solids **3**, 156–166 (1955).

If a beam made of a perfectly plastic material is subjected to combine bending and twisting, there will be certain limiting combinations of moment and torque which the beam can support. Handelman [Quart. Appl. Math. **1**, 351–353 (1944); MR **5**, 252] and Hill [Quart. J. Mech. Appl. Math. **1**, 18–28 (1948); MR **9**, 635] had previously derived a non-linear differential equation for determining this condition, and Hill and Siebel [J. Mech. Phys. Solids **1**, 207–214 (1953)] have obtained upper and lower bounds using the concepts of limit analysis. The present paper is concerned with an exact numerical solution of this equation for a square section. Relaxation methods were used, using both a 9×9 and a 13×13 mesh. Two specific values of the ratio of rate-of-twist to rate-of-bending were assumed, and the results are presented in several curves showing the stress distributions throughout the section. The limit relation between moment and torque is compared with upper and lower bounds and found to lie comfortably between them.

P. G. Hodge (Brooklyn, N. Y.).

Storchi, Edoardo. Sull'equazione fondamentale della plasticità piana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 694–713 (1953).

In the classical theory of plane plastic flow the equations of equilibrium and the yield condition furnish a non-linear differential equation for the stress function $\chi(x, y)$, the plane of flow being the plane of (x, y). A simple transformation ($u=ix+y$, $v=x+iy$ and $z=\partial\chi/\partial v$) enables the author to reduce this equation to the quasi-linear equation of Monge-Ampère:

$$\left(\frac{\partial z}{\partial v}\right)^2 \frac{\partial^2 z}{\partial u^2} - k^2 \frac{\partial^2 z}{\partial v^2} = 0,$$

where k is a constant. A first integral of this equation is then used to find a solution of the stress problem which contains an arbitrary function and an arbitrary constant. Finally this equation is transformed into the telegraph equation by integrating two of its characteristic equations and by a change of variables. The last result is well known in the theory of plasticity. The author's contribution is that no geometrical intuition is used in his derivation of this result.

E. T. Onat (Ankara).

Kostyuk, A. G. Computation of the profile of a rotating disk for conditions of creep. Prikl. Mat. Meh. **17**, 615–618 (1953). (Russian)

The radial variation in thickness of a thin rotating disk under conditions of steady creep is examined for a number of cases in which special assumptions are made on the strain-rate intensity. Integration of the equilibrium equation for the rotating disk of variable thickness yields an expression for thickness as function of the radial coordinate, the angular velocity and the two stress components. The compatibility condition for the strain velocities plus the flow law combine to give a relation between the stresses. By assuming variously that the strain rate intensity is a constant, is proportional to a power of the radial coordinate, or is proportional to a power of the stress intensity, formulas are obtained which (with considerable effort for the last two cases) could yield numerical values for the thickness variation.

W. Nachbar (Seattle, Wash.).

Hardtwig, E. Oberflächenwellen in energieverzehrenden Medien. Ann. Geofis. **7**, 143–193 (1954).

Der Kern dieser Arbeit ist die Untersuchung von Oberflächenwellen in einem visko-elastischen Voigt-Material mit der für kleine Verschiebungen gültigen Materialgleichung

$$p_{ik}=\lambda e_j^j \delta_{ik}+2\mu e_{ik}+\frac{\partial}{\partial t}[\lambda' e_j^j \delta_{ik}+2\mu' e_{ik}]$$

(p_{ik}=Spannungstensor, e_{ik}=Verzerrungstensor). Die Bestimmung von Phasengeschwindigkeit und Dämpfung führt auf eine Gleichung 9. Grades. Diese wird im Spezialfall $\lambda/\lambda'=\mu/\mu'=\tau$ formal auf die bekannte Raleighsche kubische Gleichung zurückgeführt. Für diesen Spezialfall wird der Zusammenhang zwischen Phasengeschwindigkeit, Dämpfung, Frequenz und Relaxationszeit τ im Detail untersucht, und die Bedeutung der so gewonnenen Ergebnisse für Erdbebenwellen diskutiert.

In einem vorangehenden Kapitel behandelt der Autor Oberflächenwellen in einem "Material mit Reibung" nach B. Galitzin. Nach Ansicht des Referenten ist schon die diesem Kapitel zugrunde liegende Differentialgleichung (20) sinnlos, da sie nicht invariant gegenüber Galilei-Transformationen ist, also einem Grundprinzip der klassischen Mechanik widerspricht.

W. Noll (Berlin).

Leibfried, G. Verteilung von Versetzungen im statischen Gleichgewicht. Z. Physik **130**, 214–226 (1951).

J. D. Eshelby, F. C. Frank und F. R. N. Nabarro [Phil. Mag. (7) **42**, 351–364 (1951); MR **12**, 882] haben die Verteilungen untersucht, welche parallele Versetzungen gleichen Vorzeichens in ihrer Gleitebene unter dem Einfluss einer äusseren Schubspannung annehmen können. Das dabei auftretende mathematische Problem ist zu einem Problem der Elektrostatik, das schon von T. J. Stieltjes [Acta Math. **6**, 319–324 (1885), p. 321] gelöst wurde, vollkommen analog. Vorliegende Arbeit ist eigentlich eine Fortsetzung der Untersuchungen der genannten Verfasser und behandelt die Versetzungen gleichen Vorzeichens in einem Intervall, solche verschiedenen Vorzeichens in zwei getrennten und einem gemeinsamen Intervall und ausserdem eine periodische Anordnung der Intervalle.

Das mathematische Problem ist die Lösung einer Integralgleichung vom Typ

$$A\int \frac{D(x)dx}{\xi-x} = -\tau_a(\xi), \tag{1}$$

wo $D(x)dx$ die Zahl der Versetzungen im Intervall dx, x die Gleitrichtung, A eine Konstante die für Stufen- und Schraubenversetzungen verschiedene Werte hat und $-\tau_a$ die äussere Schubspannung bedeutet. (1) folgt aus der Bedingung, dass im Gleichgewicht die gesamte Schubspannung an den Stellen verschwindet, wo die Versetzungsdichte von Null verschieden ist. Die Integralgleichung (1) wird für die erwähnten Probleme gelöst. Durch Integration über die Verteilung $D(x)$ erhält man dann die Zahl der Versetzungen, die man z.B. bei dem Vorhandensein von positiven und negativen Versetzungen mit Hilfe der vollständigen elliptischen Integrale 1. bzw. 2. Gattung ausdrücken kann. Die Schubspannungsverteilung im Aussenraum wird ebenfalls berechnet.

T. Neugebauer.

Eshelby, J. D. The equation of motion of a dislocation. Phys. Rev. (2) **90**, 248–255 (1953).

Das elastische Spannungsfeld, das eine sich bewegende reine Schraubenversetzung umgibt, wird unter der Annahme

untersucht, dass dabei nur unbedeutende Energieverluste auftreten. Den Ausgangspunkt der Berechnungen bildet die vom Verfasser [Proc. Roy. Soc. London. Ser. A. 197, 396–416 (1949)] für das eine Schraubenversetzung, welche entlang der X-Achse oszilliert, umgebende Deformationsfeld hergeleitete Formel

$$w=\tfrac{1}{4}bikH_1^{(2)}(kr)\sin\theta\, e^{i\omega t},$$

wo $k=\omega/c$ ist, b den Burgersschen Vektor der Versetzung, c die Ausbreitungsgeschwindigkeit der transversalen elastischen Wellen und $H_1^{(2)}$ eine Besselsche (Hankelsche) Funktion bedeuten. Der Mittelpunkt der Versetzung ist durch $\xi=le^{i\omega t}$ gegeben. Aus w erhält man $\dot{w}$ und, was noch wichtiger ist, die Spannungen $p_{xz}=\mu\partial w/\partial x$ und $p_{yz}=\mu\partial w/\partial y$ (μ ist der Scherungsmodul). Im dritten Teil wird ein elektromagnetisches Analogon dieses Problems besprochen. Die sich bewegende Versetzung wird durch eine sich bewegende linienartige Ladungsverteilung ersetzt, dann kann man $\dot{w}=H_x/\sqrt{\rho}$, $p_{xz}=-E_y\sqrt{\mu}$ und $p_{yz}=E_x\sqrt{\mu}$ (in Heavisideschen Einheiten) setzen. Im vierten Teil wird die Brauchbarkeit dieser Methode für den Fall der Berechnung der Bewegung einer Versetzung mit konstanter eigener Beschleunigung gezeigt und das zur Aufrechterhaltung dieser Bewegung nötige Feld wird berechnet. Im fünften Teil wird die allgemeine Bewegungsgleichung für ein vereinfachtes Modell hergeleitet und im sechsten werden die von diesem Modell herrührenden Voraussagen besprochen.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Esnault-Pelterie, Robert. Analyse dimensionnelle et métrologie. Soc. Roy. Belge Ingénieurs Industriels. Mém. no. 4 (1951), 23 pp. (1952).

In essence the article presents a clear review of the basis of dimensional analysis. To determine the needed minimum of fundamental dimensions for any specified field of physics, the author refers back to the systematic reduction of physical equations to relations between dimensionless products of physical quantities, credited to Vachy in France. He then reviews the various systems of electromagnetic units. [A very similar treatment is found in section 3 of Handbook of engineering fundamentals, edited by O. W. Eshbach, Wiley, New York, 1936, 2nd ed. 1952.] *E. Weber.*

Altenburg, Kurt, und Kästner, Siegfried. Wellenausbreitung in geschichteten Medien bei senkrechtem Einfall und die Anwendung auf Leitungstheorie, elektrische Wellen, Optik, Akustik, Wellenmachanik sowie mechanische und elektrische Vierpolketten. Ann. Physik (6) 13, 1–43 (1953).

The object of this paper is to formulate a unified mathematical scheme for treating one-dimensional linear wave-propagation phenomena covering various fields of physics. The scope of this work, however, is limited to problems which involve constant elements such as resistances, inductances, refractive indices, potential, etc. A list of the differential equations governing electrical, acoustical, optical and mechanical phenomena and the corresponding various relationships for the propagation constants, impedances and solutions of the equations are given in table I. Applying matrix theory, the author has derived relationships between the incident (normal), reflected and transmitted fields for continuous and discrete systems, including the case of a stratified medium composed of n different layers, including expressions for the propagation constant, reflection and transmission coefficients. The general results are applied to already solved problems in electric circuit theory, acoustics, mechanical systems with distributive mass, stratified optical media and the so-called tunnel effect in quantum mechanics. The paper concluded with a list of references drawn from various fields of physics and engineering. *N. Chako.*

***Turski, S., unter Mitarbeit von J. Bonder, S. Drobot, J. G. Mikusiński, W. Nowacki, J. Nowinski, W. Olszak, P. Szulkin. Die mathematischen Methoden der modernen Technik.** Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, pp. 111–125. Deutscher Verlag der Wissenschaften, Berlin, 1954.

Optics, Electromagnetic Theory

Montel, Marc. Aberrations du premier ordre des systèmes catoptriques asymétriques. Application au microscope X à réflexion totale. Opt. Acta 1, 117–126 (1954).

Geometrical and algebraic conditions are described for the correction of astigmatism both on the axis and in the field for systems composed of two mirrors. Several practical methods of correction are suggested and applications in certain special cases are discussed. *E. W. Marchand.*

Montel, Marc. Extensions de la classification des aberrations géométriques des systèmes asymétriques. Rev. Opt. 33, 585–589 (1954).

In a previous paper [Rev. Opt. 32, 585–600 (1953); MR 15, 374] the same author gave a general classification of the geometrical aberrations of asymmetric optical systems. Two assumptions were made there: infinitely small transversal aberrations of second order and a fixed position of the exit pupil. Having met with practical cases in which those conditions are not satisfied, the author has now extended his previous classification, with slight modifications, so as to cover the new examples, the theorem of Gouy being used as before. *E. W. Marchand* (Rochester, N. Y.).

Glaser, Walter, und Braun, Günther. Zur wellenmechanischen Theorie der elektronenoptischen Abbildung. I. Acta Phys. Austriaca 9, 41–74 (1954).

In wave mechanics the steady state-electron flow is represented by the function $\psi(x, y, z)$ satisfying Schrödinger's equation. This article considers the boundary-value problem of solving Schrödinger's equation when the values of $\psi(x, y, z)$ are prescribed in a plane $z=z_0$. Using optical analogy, $\psi(x, y, z)$ is derived from its values in the $z=z_0$ plane by means of Green's integral theorem (the generalized Kirchhoff formula). Series expansions for the corresponding Green's functions are derived on the basis of the reflection principle and of a treatment analogous to the WKB method. The series are asymptotic for $\hbar\to 0$. The method of stationary phase is used to evaluate the resulting diffraction integrals. The basic theorems of the method are stated, but proof is deferred to a forthcoming paper. The section dealing with the application of the method presents the actual calculation, a discussion of the number of solutions, and the geometric-optical meaning of the points of stationary phase. The solution of the diffraction problem by means of series asymptotic for $\hbar\to 0$ (the semi-classical solution) is shown to represent the connection between geometrical and wave-

mechanical electron optics. The physical meaning of the method of stationary phase is also discussed.

J. E. Rosenthal (Passaic, N. J.).

Durand, Emile. Une présentation simple de la théorie générale des systèmes de révolution en optique électronique. (Relativité et aberrations comprises.) Rev. Opt. **33**, 617–629 (1954).

This paper attacks the problem of aberrations in electron optics from the point of view of electron ballistics, disregarding both optical analogies and the eiconal method. The treatment is carried out for the most general case possible, i. e. electron trajectories are determined when the electrons are subjected to both electric and magnetic fields with both object and image in the field, the velocities being such as to require relativistic correction. The method used is essentially a perturbation method. The axially symmetrical potential is expanded in powers of r, the successive terms in this expansion being used at successive stages of approximation. The derivation of the formulas is very elegant. The calculation is greatly simplified by extending to 3 dimensions the use of complex variables in Lagrange's equations of motion originally introduced for 2-dimensional problems. [See J. E. Rosenthal, Proc. I. R. E. **39**, 10–15 (1951).]

J. E. Rosenthal (Passaic, N. J.).

Durand, Emile. Solutions générales des équations de Maxwell. C. R. Acad. Sci. Paris **236**, 1407–1409 (1953).

For any two vectors $\boldsymbol{A}$, $\boldsymbol{B}$, functions of position and time, the author establishes the following identity:

$$4\pi\boldsymbol{A}(x_1,x_2,x_3,t)=\operatorname{rot}\left\{\int_v[\operatorname{rot}'\boldsymbol{A}\pm q\boldsymbol{B}]_\tau\frac{dv'}{r}-\int_s[\boldsymbol{n}\times\boldsymbol{A}]_\tau\frac{ds'}{r}\right\}$$
$$-\operatorname{grad}\left\{\int_v[\operatorname{div}'\boldsymbol{A}]_\tau\frac{dv'}{r}-\int_s[\boldsymbol{n}\cdot\boldsymbol{A}]_\tau\frac{ds'}{r}\right\}$$
$$+q\left\{\int_v[q\boldsymbol{A}-(\pm)\operatorname{rot}'\boldsymbol{B}]_\tau\frac{dv'}{r}\pm\int_s[\boldsymbol{n}\times\boldsymbol{B}]_\tau\frac{ds'}{r}\right\}.$$

Here v is a volume containing the point x, s its bounding surface, $\boldsymbol{n}$ the outward unit normal to s, q is the operator $w^{-1}\partial/\partial t$ (w any constant), and the subscript τ indicates retarded values. [The distribution of primes follows the notation of the paper.] In this identity he first replaces $\boldsymbol{A}$ by $\boldsymbol{E}$ and $\boldsymbol{B}$ by $(\mu/\epsilon)^{1/2}\boldsymbol{H}$, taking the + sign; secondly, he replaces $\boldsymbol{A}$ by $\boldsymbol{H}$ and $\boldsymbol{B}$ by $(\epsilon/\mu)^{1/2}\boldsymbol{E}$, taking the − sign. With $w=(\epsilon\mu)^{1/2}$, this gives for $\boldsymbol{E}$ and $\boldsymbol{H}$ the following expressions in terms of vector potentials $\boldsymbol{A}$, $\boldsymbol{A}'$ and scalar potentials V, V':

$$\boldsymbol{E}=-\epsilon^{-1}\operatorname{rot}\boldsymbol{A}'-\operatorname{grad}V-\partial\boldsymbol{A}/\partial t,$$
$$\boldsymbol{H}=\mu^{-1}\operatorname{rot}\boldsymbol{A}-\operatorname{grad}V'-\partial\boldsymbol{A}'/\partial t.$$

By virtue of Maxwell's equations for a medium with constants ϵ, μ, these potentials are expressible in terms of volume integrals of current and density and surface integrals of $\boldsymbol{E}$ and $\boldsymbol{H}$; for example,

$$V=\frac{1}{4\pi\epsilon}\int_v[\rho]_\tau\frac{dv'}{r}-\frac{1}{4\pi}\int_s(\boldsymbol{n}\cdot\boldsymbol{E})_\tau\frac{ds'}{r}.$$

J. L. Synge (Dublin).

Pratelli, Aldo M. Principi variazionali del campo elettromagnetico. Ann. Scuola Norm. Sup. Pisa (3) **7** (1953), 161–203 (1954).

The basic idea behind this paper is the decomposition of a vector field into an irrotational part and a solenoidal part, and B. Finzi's generalisation of this to the similar decomposition of a skew-symmetric tensor field [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 378–382, 477–480; **13**, 211–215 (1952); MR **15**, 665, 666]. The argument is systematically presented, starting with variational principles for electrostatic and electromagnetic fields, and proceeding through Maxwellian fields to neutral meson fields, scalar and vector. The method of decomposition is extended to a 5-dimensional manifold with line-element

$$ds^2=dx_0^2-dx_1^2-dx_2^2-dx_3^2+dx_5^2,$$

a vector field being decomposed by the formula

$$H_\alpha=-\phi_\alpha+\frac{1}{3!}\epsilon_{\alpha\beta\gamma\delta\rho}\Psi^{\gamma\delta\rho|\beta},$$

and a skew-symmetric tensor field by

$$H_{\alpha\beta}=\Phi_{\beta|\alpha}-\Phi_{\alpha|\beta}+\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta\rho}\Psi^{\delta\rho|\gamma}\quad(\Phi_\alpha{}^{|\alpha}=0).$$

This is applied to scalar and vector meson fields. In the latter case one starts with a skew-symmetric tensor field $H_{\alpha\beta}$ and gives a stationary value to a quintuple integral with integrand $H_{\alpha\beta}H^{\alpha\beta}$; this leads to the equations

$$\operatorname{rot}H_{\alpha\beta}\equiv\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta\rho}H_{\alpha\beta|\gamma}=0,\quad\operatorname{div}H_{\alpha\beta}\equiv H_{\alpha\beta}{}^{|\beta}=0,$$

or equivalently

$$\Phi_\alpha{}^{|\beta}_{\beta}=0,\quad\Psi^{\delta\rho}{}^{|\beta}_{\beta}=0.$$

J. L. Synge (Dublin).

Osterberg, Harold, and Page, Nancy E. Linear recurrence equations for analyzing multilayers. J. Opt. Soc. Amer. **43**, 728–732 (1953).

In this paper the problem of propagation of plane waves incident normally on a system of parallel layers is analyzed. Each layer v is specified by its refractive index ϵ_v, permeability μ_v and thickness t_v, which are assumed to be constants for each layer. By introducing certain quantities (often used in optics), i.e. reflectances R_v, admittances S_v and transmittances T_v which are associated with the vth layer of the system, the authors have derived relations involving successive values for each set. If A_v denotes the value of either R_v, S_v or T_v for the vth layer, these relations read as follows:

$$A_{v-1}=\frac{a_vA_v+b_v}{c_vA_v+d_v}=\frac{f_v}{g_v},$$

where $a_v,\cdots,d_v$ depend on ϵ_v, μ_v and t_v. Introducing a function $Q_v=\prod_v^{v+1}g_v$, the authors have established a recurrence relation between three contiguous Q's. With the aid of the recurrence relations the quantities R_v, S_v and T_v are expressed in simple rational form in terms of Q_{v+1} and Q_{v+2}. It is pointed out that the advantage of expressing the reflectances, etc. in terms of the functions Q_v over that of the usual method is due to the existence of a recurrence relation for these quantities. *N. Chako* (New York, N. Y.).

Il'in, V. A. Problems of electrodynamics of imperfectly conducting bodies having sharp edges. Dokl. Akad. Nauk SSSR (N.S.) **97**, 213–216 (1954). (Russian)

The author investigates the effect of large but finite conductivity on the diffraction by an infinite cylinder with a finite number of sharp edges (e.g., a prism). The boundary condition used is of the "skin effect" type, i.e., a linear homogeneous boundary condition with a complex ratio of the Hertz potential to its normal derivative. This complex ratio is expanded in a power series in w, the resistivity of the

cylinder. Assuming that the solution for the perfectly conducting cylinder of the same shape is known, the author calculates the solution in the imperfectly conducting case, retaining terms in w and w^2.

In this calculation it turns out that the first-order term neglects the curvature of the cylinder, while in the second term the integrand becomes infinite at corners. The author smooths out all corners, replacing them by arcs of finite curvature. He then lets this smoothed boundary approach the original one. He finds that in the limit there is an additional contribution in w^2 from each edge. This extra term has logarithmic behavior near the edge, thus dominating the behavior of the field, other terms being finite.

J. Shmoys (New York, N. Y.).

*Il'in, V. A. **Electrodynamic problems for nonideal conducting bodies having angular lines.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 5 pp. (mimeographed). $2.50.

Translation of the paper reviewed above.

*Hesselbach, Benno. **Gekrümmte Hohlleiter.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 251–253. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

For a hollow tube of arbitrary but uniform cross-section, bent with radius of curvature R, the author deduces the general differential equation of the electromagnetic field in terms of the complex vector $\mathbf{F}=\mathbf{E}+i\mathbf{H}$, using bicomplex notation and assuming the time dependence as $e^{i\omega t}$. It is shown that the result is a two-dimensional boundary-value problem, separable in the four components of the field vectors in the meridianal plane. *E. Weber.*

Nadjakoff, Emil G. Ein zweidimensionales elektrostatisches Problem. I. C. R. Acad. Bulgare Sci. **6** (1953), no. 2, 1–4 (1954). (Russian summary)

Nadjakoff, Emil G. Ein zweidimensionales elektrostatisches Problem. II. C. R. Acad. Bulgare Sci. **6** (1953), no. 2, 5–8 (1954). (Russian summary)

The author calculates the capacitance coefficients of a system consisting of a half-plane and of four right-angle wedges arranged like the corners of two streets which intersect at right angles. The half-plane is in the plane of symmetry of the configuration of the four wedges. The problem is solved by means of a Schwarz-Christoffel mapping. Approximate expressions for the capacitance coefficients are given for the case when the distance between the edge of the half-plane and the "street intersection" is very large compared to both "street" widths. In the second part of the paper the geometry is further complicated by the introduction of a second half-plane down the center of the same street as the first half-plane but from the opposite end. The problem arose in the theory of the quadrant electrometer.

J. Shmoys (New York, N. Y.).

v. Zweygbergk, Svante. **Die Reaktanz der Bruchlochwicklungen.** Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. **175**, 25 pp. (1954).

The reactances of fractional slot windings are calculated by tearing the windings into isolated conductors and interconnecting the individual reactances with the aid of a matrix of transformation. The usual Fourier analysis of flux-waves produced by the windings is thereby circumvented.

G. Kron (Schenectady, N. Y.).

Mishkin, E. Theory of the squirrel-cage induction machine derived directly from Maxwell's field equations. Quart. J. Mech. Appl. Math. **7**, 472–487 (1954).

To simplify the mathematical formulation, the author creates a model of the induction machine by developing the cylindrical structure into a one of parallel planes. The stator and rotor yokes are taken to be infinitely permeable, whereas the teeth areas are assumed to be magnetically anisotropic, having high permeability normal to, and low permeability parallel to, the air gap. In addition, the stator tooth area carries adjacent to the airgap a thin layer of axial current flow with transverse periodicity defined by the pole pitch; and the rotor tooth area has an equivalent axial electrical conductivity to simulate the conductors of the squirrel cage embedded in the rotor slots.

The relative motion of the rotor is taken into account by a moving system of references axes and by stipulating a rotor frequency $s\omega$, where s is the "slip". The complete solution of Maxwell's equations, disregarding the displacement current, leads to an expression for the stator current which includes the skin effect as a function of rotor frequency or slip. Comparison with the classical diagrams reveals significant deviations of current and torque.

E. Weber (Brooklyn, N. Y.).

Mishkin, E. Disturbances in the induction machine due to broken squirrel-cage rings. J. Franklin Inst. **259**, 133–143 (1955).

Based upon a previous paper [Dissertation, Haifa, 1951], the author examines the effect of peripheral asymmetry by stipulating equivalent local impressed voltages in the end rings of the squirrel cage. The various harmonic currents are computed indicating parasitic torques leading to saddle effects in the torque-speed curve. *E. Weber.*

Beghin, Bernard. L'équation matricielle des lignes exponentielles sans pertes. C. R. Acad. Sci. Paris **240**, 168–170 (1955).

The solution of the exponential transmission line is given by means of the matrix calculus and it is shown that it can be interpreted as the cascade connection of two fourpoles. One of these is an ideal transformer, the other represents a homogeneous transmission line with increased guided wave length. *E. Weber* (Brooklyn, N. Y.).

*Duffin, R. J. **Impossible behavior of nonlinear networks.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 124–128. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

Let the following definitions be made: (i) a resistor is quasi-linear if the voltage drop across the resistor is a function only of the current through the resistor and if the differential resistance (dv/di) is positive; (ii) a primary resistor is one which is in that part of an electrical network consisting of a simple closed circuit containing a generator but no capacitors. It is shown that an isothermal electro-mechanical system whose primary resistors are quasi-linear cannot convert direct current to alternating current. *R. Kahal.*

Ba Hli, Freddy. Network synthesis by impulse response for specified input and output in the time domain. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. **261**, i+65 pp. (1953).

The problem discussed is that of finding a network when the excitation and response functions are specified in the time domain. The input-output pair, as time functions, are

related through the convolution integral by the impulsive response of the network. When excitation and response are prescribed it is necessary, in general, to solve an integral equation to find the network impulsive response. The synthesis problem is somewhat simpler in the frequency domain since the required system function is merely the ratio of the response transform to the excitation transform. However, even when input and output are given as analytic time functions it may not be satisfactory to transform them into frequency functions, since the ratio of their transforms may not yield a realizable system function. An approximation procedure then becomes necessary and there is no guarantee that a satisfactory approximation in the frequency domain will necessarily be a good approximation in the time domain. Thus the necessity for solving the integral equation remains. The author presents a systematic numerical method for integrating the convolution integral and obtaining thereby an approximate expression for the network impulse response. Three methods are presented for obtaining the (frequency) system function as a rational fraction after the impulse response has been obtained. Standard synthesis procedures may then be used to construct the network.

R. Kahal (Monterey, Calif.).

Quantum Mechanics

Have

*von Neumann, John. **Mathematical foundations of quantum mechanics.** Translated by Robert T. Beyer. Princeton University Press, Princeton, 1955. xii+445 pp. $6.00.

Translation of the author's Mathematische Grundlagen der Quantenmechanik [Springer, Berlin, 1932].

*Schrödinger, Erwin. **The meaning of wave mechanics. La signification de la mécanique ondulatoire.** Louis de Broglie, physicien et penseur, pp. 16–32. Editions Albin Michel, Paris, 1953. 870 francs. (English and French)

Have

*Elsasser, W. M. **Les mesures et la réalité en mécanique quantique.** Louis de Broglie, physicien et penseur, pp. 87–108. Editions Albin Michel, Paris, 1953. 870 francs.

*March, A. **Mécanique ondulatoire et concept de substance.** Louis de Broglie, physicien et penseur, pp. 109–115. Editions Albin Michel, Paris, 1953. 870 francs.

*Reichenbach, Hans. **La signification philosophique du dualisme ondes-corpuscules.** Louis de Broglie, physicien et penseur, pp. 117–134. Editions Albin Michel, Paris, 1953. 870 francs.

*von Weizsaecker, C. F. **Sur le problème d'une théorie unitaire des champs.** Louis de Broglie, physicien et penseur, pp. 135–143. Editions Albin Michel, Paris, 1953. 870 francs.

*Watanabé, Satosi. **Réversibilité contre irréversibilité en physique quantique.** Louis de Broglie, physicien et penseur, pp. 385–400. Editions Albin Michel, Paris, 1953. 870 francs.

*Costa de Beauregard, Olivier. **L'irréversibilité quantique, phénomène macroscopique.** Louis de Broglie, physicien et penseur, pp. 401–412. Editions Albin Michel, Paris, 1953. 870 francs.

Gell-Mann, M., Goldberger, M. L., and Thirring, W. E. Use of causality conditions in quantum theory. Phys. Rev. (2) **95**, 1612–1627 (1954).

The paper deals with the question of deducing conditions on scattering amplitudes from causality requirements. The causality requirement is specified by the condition that the commutator of two Heisenberg operators for the field (in the case of a boson field) shall vanish if these operators are taken at two points separated by a space-like interval. This describes the fact that, since disturbances cannot propagate with a speed greater than that of light, the measurements of two observable quantities at points with a space-like separation should not interfere with each other. The problems of the scattering of zero-spin particles by a rigid sphere and the scattering of photons by a quantized matter field are discussed, and the dispersion relation of R. Kronig [J. Opt. Soc. Amer. **12**, 547–557 (1926)] and H. A. Kramers [Atti Congresso Internaz. Fis., Como-Pavia-Roma, 1927, v. 2, Zanichelli, Bologna, 1928, pp. 545–557] is obtained. Applications are made to the nuclear photoeffect and to the scattering of gamma rays by protons. *N. Rosen* (Haifa).

Bohm, David. Proof that probability density approaches $|\psi|^2$ in causal interpretation of the quantum theory. Phys. Rev. (2) **89**, 458–466 (1953).

In previous papers [Phys. Rev. (2) **85**, 166–179, 180–193 (1952); MR **13**, 709, 710] the author developed a causal interpretation of the quantum theory. This included the idea of a statistical ensemble of particles moving classically under the action of a certain force, the probability density being $P=|\psi|^2$, where ψ is the Schroedinger wave function. In the present paper it is shown that, if one starts with an arbitrary probability density P, in the course of time it will approach $|\psi|^2$ as a result of random collisions.

N. Rosen (Haifa).

Have

*Bohm, David. **Proof that probability density approaches $|\psi|^2$ in causal interpretation of the quantum theory.** New research techniques in physics, pp. 187–198. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Cientifica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

This is essentially a shorter version of the preceding paper.

N. Rosen (Haifa).

Bohm, D., and Vigier, J. P. Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations. Phys. Rev. (2) **96**, 208–216 (1954).

The authors propose a physical model conforming to the causal interpretation of the quantum theory [see the two preceding reviews]. This consists of a fluid of density R^2 and mean velocity $\nabla S/m$, where $\psi=Re^{iS/\hbar}$, ψ being the Schroedinger wave function of a particle of mass m. The fluid undergoes random fluctuations in its motion. If a particle is represented by a small inhomogeneity in this fluid, it is shown that the probability density of particles P in a statistical ensemble approaches $|\psi|^2$ in the course of time.

N. Rosen (Haifa).

Demkov, Yu. N. The principle of detailed balance in quantum mechanics and some identities for the amplitudes of scattering in the theory of collisions. Dokl. Akad. Nauk SSSR. (N.S.) **97**, 1003–1006 (1954). (Russian)

Relations are found between the scattering amplitudes for associated pairs of directions of incidence and scattering

which can be interpreted on the basis of the principle of detailed balance. *N. Rosen* (Haifa).

Gregory, Christopher. Conservation of quanta for nonlocal fields. Phys. Rev. (2) **89**, 1199 (1953).

The author gives a definition of spatial and time-like integration of non-local quantities. This definition is applied to the computation of the momentum-energy vector from the stress-energy tensor and an invariant from a four-vector proposed as the quantum current-density sector. The theory is applied to a massless boson field and it is shown that only the energy is positive definite while the operator for the number of quanta is not. *A. H. Taub.*

Ôno, Yôrô. On the constants of motion for the case of non-localized interactions. Progr. Theoret. Phys. **10**, 125–136 (1953).

The author considers a spinor field and a neutral scalar field with a non-local interaction between them. A general form of the Lagrangean function is assumed and a stress-energy tensor is derived. It is shown that the latter tensor satisfies four continuity equations as a consequence of the field equations. By space integration of the μ^4 components of the stress-energy tensor, the energy-momentum vector is obtained, and shown to be a constant of the motion in consequence of the continuity equations. A current four-vector is also constructed. The invariance of the Lagrangean function under general transformations of coordinates and under gauge transformation is used to derive the appropriate stress-energy tensor and current vector as is done in general relativity. *A. H. Taub* (Urbana, Ill.).

Kahan, T., et Rideau, G. Sur la déduction de divers principes variationnels de la théorie des collisions à partir d'un principe unique. J. Phys. Radium (8) **13**, 326–332 (1952).

The authors point out that if L is a linear symmetric operator then the solutions of the equations $L\phi=0$ are extremals of the integral $I=\int\psi_2 L\psi_1 dt$. They apply this result to obtain a unified variational formulation of various problems arising in the quantum-mechanical treatment of collisions of particles. *A. H. Taub* (Urbana, Ill.).

Królikowski, W. Connections between the Einstein-Infeld approximation method and the perturbation method. Bull. Acad. Polon. Sci. Cl. III. **2**, 375–377 (1954).

The author derives a formal series solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=(\overset{0}{H}+\overset{1}{H})\Psi$$

by using a representation where this equation becomes

$$i\hbar\frac{\partial\Psi_B}{\partial t}=\overset{0}{H}_B(t)\Psi_B.$$

He writes $\Psi_B=u_B(t)e^{-iEt/\hbar}$ and $u_B=\sum_{l=0}^{\infty}u_B^{(l)}$. Relations between the time derivative of $u^{(l)}$ and $u^{(l+1)}$ are used in the formal solution referred to. *A. H. Taub* (Urbana, Ill.).

Moshinsky, Marcos. Forces tensorielles dépendant de la vitesse. J. Phys. Radium (8) **15**, 725–732 (1954).

For a two-particle system, the most general form of non-relativistic potential depending only on the relative position and momentum and on the spins is written down and is transformed to a vector-scalar (i.e., triplet-singlet) representation. The form is restricted by symmetry considerations and is finally chosen to be quadratic in the momentum. For a "square-well potential" dependence on the radius, the Schroedinger equation for the triplet wave functions becomes identical with the equations of elasticity. Solutions are expressed in terms of vector spherical harmonics. From the expressions for probability density and current density boundary conditions are set up for joining solutions at the discontinuity of the potential. *N. Rosen* (Haifa).

***Moshinsky, Marcos. Diffraction in time.** New research techniques in physics, pp. 201–215. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

A totally absorbing shutter in a one-dimensional stream is opened instantaneously. The resulting transient effect is discussed for the stream obeying (a) the Schroedinger equation, (b) the wave equation, (c) the Klein-Gordon equation. In case (a), the rise to and oscillation about the final steady state are expressed explicitly in terms of Fresnel's integrals. The solution is given diagrammatic representation by means of the Cornu spiral occurring in diffraction theory, hence the title of the paper. Case (b) is of little interest. The general solution for (c) is given and it is shown that in the limit as the speed of light approaches zero this reduces to (a).
A. J. Coleman (Toronto, Ont.).

***Wataghin, G. On a statistical theory of non-local fields and on the production of fundamental particles.** New research techniques in physics, pp. 267–274. Symposium organized by the Academia Brasileira de Ciências and Centre de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

A relativistically invariant method of introducing a cut-off in the form factor for interaction between particles is described and the possibility is suggested that the form factor determines the energy distribution of particles created in high-energy collisions. *A. J. Coleman* (Toronto, Ont.).

***Schützer, Walter, and Tiomno, J. On the connection of the scattering matrix with causality.** New research techniques in physics, pp. 281–284. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

The same authors [Phys. Rev. (2) **83**, 249–251 (1951); MR **13**, 194] showed that the causality condition implies that for a single non-relativistic spinless particle of zero angular momentum the poles of the scattering matrix $S(k)$ are either pure imaginary or in the lower half of the k-plane. This result is here extended to particles of arbitrary angular momentum. *A. J. Coleman* (Toronto, Ont.).

Wick, G. C. Properties of Bethe-Salpeter wave functions. Phys. Rev. (2) **96**, 1124–1134 (1954).

The four-dimensional wave function $\psi(1, 2)$ representing the proton, neutron system in a bound state is defined as $\psi(1, 2)=\langle 0|T\psi_P(x_1)\psi_N(x_2)|P\rangle$, where T is the chronological

ordering operator. Using the fact that the rest mass E of the bound state ($E=(P^2)^{1/2}$), is less than the combined rest masses of the two free nucleons, the author shows that the function $\chi(x_1-x_2)$ defined as $\psi(1,2)\exp[-\frac{1}{2}iP(x_1+x_2)]$ contains only positive (negative) frequencies for positive (negative) values of the relative time variable (t_1-t_2). This important remark makes it possible to continue the function χ analytically to complex values of (t_1-t_2), and in particular to purely imaginary values. A similar continuation is possible in momentum space in the complex p_0-plane, where a wave equation satisfied by $\chi(p)$ with real $p_4=ip_0$ is obtained by a rotation of the integration path. This equation is formally the same as that proposed by Salpeter and Bethe [Phys. Rev. (2) **84**, 1232–1242 (1951); MR **14**, 707] except that it now acquires an "elliptic" rather than a "hyperbolic" metric.

The equation is particularly considered for the simple case of two spinless particles bound by a scalar field of zero rest mass. In the "ladder" approximation an integral representation method is presented which reduces the eigenvalue problem to the standard Sturm-Liouville type.

A. Salam (Cambridge, England).

Cutkosky, R. E. Solutions of a Bethe-Salpeter equation. Phys. Rev. (2) **96**, 1135–1141 (1954).

This paper continues the discussion of the eigenvalue problem for the simplified Bethe-Salpeter equation [see the preceding review]. Two distinct methods are employed to solve the problem: (a) An adaptation of the method used previously by Fock [Z. Physik **98**, 145–154 (1935)] to solve the hydrogen atom problem. This consists of mapping the four-dimensional momentum space onto the surface of a five-dimensional sphere by a stereographic projection. (b) The Wick transform method [see the preceding review]. In addition to the solutions with the expected non-relativistic limit, the limit of a non-relativistic hydrogen atom, a large number of anomalous solutions are obtained if the interaction constant is large. These do not correspond to any known physical situation. *A. Salam.*

Gotô, Ken-iti. Wave equations with new degree of freedom. Progr. Theoret. Phys. **12**, 208–224 (1954).

The author considers in detail the structure of wave equations of the type introduced by Pais [same journal **10**, 457–469 (1953)]. These equations are invariant for rotations in space-time as well as in the isotopic spin space (w-space), and describe elementary particles with arbitrary spin values S and arbitrary w-spin values T. The formalism employed to introduce the new degrees of freedom of the w-space is an extension of that used by Dirac [Proc. Roy. Soc. London. Ser. A. **155**, 447–459 (1936)] and Fierz [Helv. Phys. Acta **12**, 3–37 (1939)]. The final sections make some ingenious suggestions regarding the possibilities of space-time approach to the new degrees of freedom, in particular of their connection with curved space-time. *A. Salam.*

Rivier, Dominique. On the quantum theory of fields. Progr. Theoret. Phys. **9**, 633–662 (1953).

The author proposes a new method of quantization for Lorentz and gauge invariant fields, making no use of the (non-covariant) canonical formalism. The principle of the method is the following. To the various infinitesimal transformations of the Lorentz and gauge groups correspond constants of the motion which are deducible from the lagrangian up to the ordering of factors. The commutators of these constants of the motion with the field quantities are completely determined by the structure of the Lorentz and gauge groups. The quantization rule is to deduce from the above mentioned commutators the commutation rules for field quantities, including the choice of the statistics (commutators versus anticommutators), as well as all orderings of factors. The method is applied to free fields and leads to the conventional results. *L. Van Hove* (Utrecht).

Brachman, Malcolm K. Space-time representation in wave mechanics: illustration of the method. Phys. Rev. (2) **96**, 516–518 (1954).

See E. J. Hellund and M. K. Brachman, Phys. Rev. (2) **92**, 822–824 (1953) [MR **15**, 380]. The procedure developed there is applied to the Schroedinger equation for a square-well potential. *N. Rosen* (Haifa).

Hozyainov, V. T. On the theory of collisions of particles with given angular momentum. Ž. Eksper. Teoret. Fiz. **27**, 275–282 (1954). (Russian)

In order to carry out calculations for processes involving a system of two fermions with given angular momentum, relativistic wave functions expressed in terms of spherical vector harmonic functions are found for the system, and these are expanded in plane waves. *N. Rosen* (Haifa).

Case, K. M. Some generalizations of the Foldy-Wouthuysen transformation. Phys. Rev. (2) **95**, 1323–1328 (1954).

The equation given by M. Taketani and S. Sakata [Proc. Phys.-Math. Soc. Japan **22**, 757–770 (1940)] is used to describe particles of spin 0 and 1. To separate the positive and negative energy states a canonical transformation is introduced, similar to that given for spin $\frac{1}{2}$ by L. L. Foldy and S. A. Wouthuysen [Phys. Rev. (2) **78**, 29–36 (1950)]. The forms of the operators obtained in this way and the non-relativistic limits are similar to those for spin $\frac{1}{2}$. The procedure is also extended to higher spins. The method is applied to the case of particles in a magnetic field.

N. Rosen (Haifa).

***Yukawa, Hideki. An attempt at a unified theory of elementary particles.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 2–6; discussion, pp. 7–12. Science Council of Japan, Tokyo, 1954. $10.00.

***Møller, Christian. On the problem of convergence in non-local field theories.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 13–17; discussion 17–23. Science Council of Japan, Tokyo, 1954. $10.00.

***Peierls, Rudolf E. Field theories with non-local interactions.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 24–29; discussion, pp. 29–39. Science Council of Japan, Tokyo, 1954. $10.00.

***Pais, Abraham. ω-space theory.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 156–161; discussion, pp. 161–163. Science Council of Japan, Tokyo, 1954. $10.00.

*Bopp, Fritz. **Über die Bedeutung der konformen Gruppe in der Theorie der Elementarteilchen.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 289–294; discussion, p. 295. Science Council of Japan, Tokyo, 1954. $10.00.

Thermodynamics, Statistical Mechanics

Fujita, Shigeichi. Criticism on the expressions of the second law of thermodynamics. Kumamoto J. Sci. Ser. A. 1, no. 4, 31–39 (1954).

The author sets out to correct inexactitudes in the statement of the Second Law in three (rather old: 1931, 1932, 1939) text-books on Thermodynamics. Although it is desirable from time to time to re-state the Second Law and to examine precisely the conditions under which its consequences are applicable, in particular whether they do, or do not, imply reversibility, it is doubtful whether the author achieved this aim. The argument hinges on the recognition that for real (irreversible) processes:

$$\delta W_1 < P\,dV \quad \text{and} \quad \delta W_2 < \sum_{i=1}^{n} X_i\,dx_i,$$

where δW_1 denotes the work done by pressure and δW_2 denotes the work done by forces other than pressure, the remaining symbols being standard. The author does not show how these statements (certainly true) are derived (presumably) from the Second Law, nor does he justify them on any other grounds (Author's eqns. (5)).

The conclusions drawn by the author are, generally, well understood. The peculiarities of the paper's style make it sometimes difficult to perceive its intentions.

J. Kestin (Providence, R. I.).

Mercier, Jean. Considérations sur l'expression du travail en thermodynamique. C. R. Acad. Sci. Paris **240**, 605–608 (1955).

Noting that the heat dQ and work $d\mathcal{T}$ appear in the well-known statement of the First Law (for a cyclic process), $dQ=d\mathcal{T}$, in a symmetrical manner, the paper endeavors to build up a series of expressions for $d\mathcal{T}=p\,dv$ which are symmetrical to the many expressions for $dQ=T\,dS$. For example, in analogy with specific heat $c=dQ/dT$ the paper introduces the concept of specific work $b=-d\mathcal{T}/dp$.

As the paper states "there is no doubt that the different coefficients which we have just defined, and which can be used for the evaluation of work, can render the same services as those rendered by the calorimetric coefficients to which we are used. It is, however, necessary to state that we cannot do without the latter, being incapable of evaluating the quantities of heat or the corresponding changes in entropy in a process without them; on the other hand we can measure both pressures and volumes. Nevertheless, in certain cases, it seems useful to have expressions for work other than pdv to represent an element of work entering a process."

J. Kestin (Providence, R. I.).

*Prigogine, I. **On some aspects of thermodynamics of irreversible processes.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 475–485; discussion, pp. 485–487. Science Council of Japan, Tokyo, 1954. $10.00.

Expository paper.

Schönberg, M. Application of second quantization methods to the classical statistical mechanics. II. Nuovo Cimento (9) **10**, 419–472 (1953).

[For part I see Nuovo Cimento (9) **9**, 1139–1182 (1952); MR **14**, 710.] A wave formalism is applied to the Liouville equation of classical mechanics which describes the time variation of probability distributions in phase space for a N-particle system. The Liouville equation is now supposed to hold for a complex-valued "wave function", and the probability distribution is now defined as the absolute square of the wave function. By assuming the latter symmetric or antisymmetric the ideas of Bose and Fermi statistics are introduced in classical theory. The formalism is then used to develop the ensemble theory of statistical mechanics. Except for the rather trivial avoidance of the Gibbs paradox all the familiar results are obtained. No physical distinction appears between Bose and Fermi particles, unless the highly artificial assumption of a subdivision of phase space in cells is introduced. At the end of the paper a new derivation of the Boltzmann equation is proposed; it lacks proper justification as far as the reviewer can see. Also the proposal to extend the Boltzmann equation to ternary collisions by means of Kirkwood's superposition hypothesis is completely unjustified. The paper contains further statements which to the reviewer were incomprehensible, like at the bottom of p. 425, where it is claimed that in a quantum theory of fields the occurrence of a S matrix different from unity is an irreversible feature. Taken with the usual interpretation of all words, this statement is of course entirely wrong.

L. Van Hove (Utrecht).

ter Haar, D., and Green, C. D. The Ehrenfests' wind-wood model in two dimensions. Proc. Cambridge Philos. Soc. **51**, 141–148 (1955).

Two stochastic models are treated. The first, corresponding to the Stosszahlensatz, yields, trivially, an exponential approach to equilibrium. The second, assuming a Bernoulli distribution of collisions, is more difficult, including fluctuations and leading to formulas for the mean life and average recurrence time of a state. *H. Grad* (New York, N. Y.).

Cox, R. T. The statistical method of Gibbs in irreversible thermodynamics. J. Phys. Chem. **56**, 1030–1033 (1952).

The law of motion of a system with quantized states f_i is assumed to be

$$\dot{f}_i=\sum_j (f_j\lambda_{ji}-f_i\lambda_{ij}) \quad (\lambda_{ij}=\lambda_{ji}).$$

With reference to a given state variable n_i, mean value $\langle n\rangle$, it is assumed that the instantaneous distribution of states is $f_i=e^{M+Nn_i}$, and sufficiently close to equilibrium we find $d\langle n\rangle/dt=g\partial S/\partial\langle n\rangle$ when $S=-k\sum f_i\log f_i$ and g is positive. For several variables n', n'', $\cdots$, a symmetric matrix $g^{\alpha\beta}$ is found. For systems in thermal contact with the surroundings,

$$(*)\quad \dot{f}_i=\sum_j (f_j e^{E_j/kT}-f_i e^{E_i/kT})\mu_{ij} \quad (\mu_{ij}=\mu_{ji}=\lambda_{ji}e^{-E_j/kT})$$

is assumed and similar conclusions found. A Rayleigh dissipation function can be introduced and also a generalized dissipation function including temperature changes as well.

If Liouville's equation is modified with a stochastic term corresponding to (*), it is stated that certain known results in the theory of Brownian motion follow. *H. Grad.*

Jauho, Pekka. A remark on the definition of entropy in Bose-Einstein and Fermi-Dirac statistics. Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. **180**, 6 pp. (1954).

Results of a previous paper [same Ann. no. **172** (1954); MR **16**, 550] are extended to apply also to quantum statistics. *G. Newell* (Providence, R. I.).

Parker, Eugene N. Tensor virial equations. Phys. Rev. (2) **96**, 1686–1689 (1954).

The classical virial equation for a system of interacting particles is the contraction of a similar tensor equation. The latter is applied to the discussion of anisotropic hydrodynamical systems and to the change with time of the size of anisotropic clusters of particles. *L. Van Hove.*

Rosenstock, Herbert B. Dynamics of simple lattices. Phys. Rev. (2) **97**, 290–303 (1955).

For the three monatomic cubic lattices and for not too large values of the ratio of 2nd nearest neighbour to 1st nearest neighbour forces, the author is able to localize and describe exactly the singularities of the elastic frequency distribution, the existence of which had been established for general crystals by the reviewer [Phys. Rev. (2) **89**, 1189–1193 (1953); MR **15**, 88]. He further modifies the approximate moment method proposed by Montroll [J. Chem. Phys. **10**, 218–229 (1942)] for the calculation of the frequency distribution in order to take due account of the singularities. This modification had been proposed independently by Lax and Lebowitz [Phys. Rev. (2) **96**, 594–598 (1954)]. *L. Van Hove* (Utrecht).

Saitô, Nobuhiko. Statistical mechanics of flexible linear polymers. I. Moments of end-to-end distance of a polymer. J. Phys. Soc. Japan **9**, 780–785 (1954).

An equation is derived giving the average square end-to-end distance of a long chain polymer of hard spheres as a sum of two terms, one proportional to N, the number of molecules, and the other term proportional to $N^{3/2}$. This result is in substantial agreement with some other approximate expressions obtained in other ways by Yamamoto and Teramoto [Busseiron Kenkyu **47**, 18 (1952)]. The approach is described in this paper as a statistical-mechanical one even though the temperature does not seem to enter in any essential way. Among the "assumptions" is one saying that if the chain is cut into three pieces, the pieces are statistically independent. This is not an assumption but an approximation made at a strategic point in the derivation, for it actually is in contradiction with his a-priori probability assumption. The implication of this is not entirely clear. *G. Newell.*

***Kirkwood, John G. The theory of visco-elastic properties of high polymers.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 388–396; discussion, pp. 396–397. Science Council of Japan, Tokyo, 1954. $10.00.

***Mayer, Joseph E. Theoretical treatment of liquids.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 415–424; discussion, pp. 425–427. Science Council of Japan, Tokyo, 1954. $10.00.

***Kirkwood, John G. The statistical mechanical theory of the liquid state.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 428–431; discussion, pp. 431–434. Science Council of Japan, Tokyo, 1954. $10.00.

***Ono, Syu. Quantum-statistical theory of transport phenomena.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 449–458; discussion, pp. 458–459. Science Council of Japan, Tokyo, 1954. $10.00.

***Prigogine, I. On the statistical mechanics of irreversible processes.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 464–470. Science Council of Japan, Tokyo, 1954. $10.00.

***Hashitsume, Natsuki. Thermal fluctuations in linear dissipative systems.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 495–503; discussion, pp. 504–506. Science Council of Japan, Tokyo, 1954. $10.00.

***de Boer, Jean. Quantum properties of the condensed state.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 507–527; discussion, 527–530. Science Council of Japan, Tokyo, 1954. $10.00.

Expository paper.

***Ikeda, Kazuyosi. On the theory of condensation.** Proceedings of the international conference of theoretical physics, Kyoto and Tokyo, 1953, pp. 544–553; discussion, p. 553. Science Council of Japan, Tokyo, 1954. $10.00.